Journal of Computational Mathematics
Vol.28, No.4, 2010, 517–551.
http://www.global-sci.org/jcm
doi:10.4208/jcm.1003-m0015

AN EFFICIENT METHOD FOR MULTIOBJECTIVE OPTIMAL CONTROL AND OPTIMAL CONTROL SUBJECT TO INTEGRAL CONSTRAINTS*

Ajeet Kumar
Department of Theoretical & Applied Mechanics, Cornell University, Ithaca, NY 14853, USA
Email: ak428@cornell.edu
Alexander Vladimirsky
Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
Email: vlad@math.cornell.edu

Abstract

We introduce a new and efficient numerical method for multicriterion optimal control and single criterion optimal control under integral constraints. The approach is based on extending the state space to include information on a "budget" remaining to satisfy each constraint; the augmented Hamilton-Jacobi-Bellman PDE is then solved numerically. The efficiency of our approach hinges on the causality in that PDE, i.e., the monotonicity of characteristic curves in one of the newly added dimensions. A semi-Lagrangian "marching" method is used to approximate the discontinuous viscosity solution efficiently. We compare this to a recently introduced "weighted sum" based algorithm for the same problem [25]. We illustrate our method using examples from flight path planning and robotic navigation in the presence of friendly and adversarial observers.

Mathematics subject classification: 90C29, 49L20, 49L25, 58E17, 65N22, 35B37, 65K05.
Key words: Optimal control, Multiobjective optimization, Pareto front, Vector dynamic programming, Hamilton-Jacobi equation, Discontinuous viscosity solution, Semi-Lagrangian discretization.

1. Introduction

In the continuous setting, deterministic optimal control problems are often studied from the point of view of dynamic programming; see, e.g., [1, 8]. A choice of the particular control $\boldsymbol{a}(t)$ determines the trajectory $\boldsymbol{y}(t)$ in the space of system-states $\Omega \subset \boldsymbol{R}^n$. A running cost K is integrated along that trajectory and the terminal cost q is added, yielding the total cost associated with this control. A *value function* u, describing the minimum cost to pay starting from each system state, is shown to be the unique viscosity solution of the corresponding Hamilton-Jacobi PDE. Once the value function has been computed, it can be used to approximate optimal feedback control. We provide an overview of this classic approach in section 2.

However, in realistic applications practitioners usually need to optimize by many different criteria simultaneously. For example, given a vehicle starting at $\boldsymbol{x} \in \Omega$ and trying to "optimally" reach some target $\mathcal{T}$, the above framework allows to find the most fuel efficient trajectories and the fastest trajectories, but these generally will not be the same. A natural first step is to compute the total time taken along the most fuel-efficient trajectory and the total amount of fuel needed to follow the fastest trajectory. Computational efficiency requires a method for

* Received January 15, 2009 / Revised version received July 13, 1009 / Accepted October 17, 2009 / Published online April 19, 2010 /

computing this simultaneously for all starting states $\boldsymbol{x}$. A PDE-based approach for this task is described in section 3.1.

This, however, does not yield answers to two more practical questions: what is the fastest trajectory from $\boldsymbol{x}$ to $\mathcal{T}$ without using more than the specified amount of fuel? Alternatively, what is the most fuel-efficient trajectory from $\boldsymbol{x}$, provided the vehicle has to reach $\mathcal{T}$ by the specified time? We will refer to such trajectories as *constrained-optimal*.

One approach to this more difficult problem is the *Pareto optimization*: finding a set of trajectories, which are optimal in the sense that no improvement in fuel-efficiency is possible without spending more time (or vice versa). This defines a *Pareto front* – a curve in a time-fuel plane, where each point corresponds to time & fuel needed along some Pareto-optimal trajectory. This approach is generally computationally expensive, especially if a Pareto front has to be found for each starting state $\boldsymbol{x}$ separately. The current state of the art for this problem has been developed by Mitchell and Sastry [25] and described in section 3.2. Their method is based on the usual *weighted sum* approach to multiobjective optimization [24]. A new running cost K is defined as a weighted average of several competing running costs K_i's, and the corresponding Hamilton-Jacobi PDE is then solved to obtain one point on the Pareto front. The coefficients in the weighted sum are then varied and the process is repeated until a solution satisfying all constraints is finally found. Aside from the computational cost, the obvious disadvantage of this approach is that only a convex part of the Pareto front can be obtained by weighted sum methods [13], which may result in selecting suboptimal trajectories. In addition, recovering the entire Pareto front for each $\boldsymbol{x} \in \Omega$ is excessive and unnecessary when the real goal is to solve the problem for a fixed list of constraints (e.g., maximum fuel or maximum time available).

Our own approach (described in section 3.3) remedies these problems by systematically constructing the exact portion of Pareto front relevant to the above constraints *for all* $\boldsymbol{x} \in \Omega$ *simultaneously*. Given $\Omega \subset \boldsymbol{R}^n$ and r additional integral constraints, we accomplish this by solving a single augmented partial differential equation on a $(r+n)$-dimensional domain. Our method has two key advantages. First, it does not rely on any assumptions about the convexity of Pareto front. Secondly, the PDE we derive has a special structure, allowing for a very efficient marching method. Our approach can be viewed as a generalization of the classic equivalency of Bolza and Mayer problems [8]. The idea of accommodating integral constraints by extending the state space is not new. It was previously used by Isaacs to derive the properties of constrained-optimal strategies for differential games [21]. More recently, it was also used in infinite-horizon control problems by Soravia [37] and Motta & Rampazzo [26] to prove the uniqueness of the (lower semi-continuous) viscosity solution to the augmented PDE. However, the above works explored the theoretical issues only and, to the best of our knowledge, ours is the first practical numerical method based on this approach. In addition, we also show the relationship between optimality under constraints and Pareto optimality for feasible trajectories.

The computational efficiency of our method is deeply related to the general difference in numerical methods for time-dependent and static first-order equations. In optimal control problems, time-dependent HJB PDEs result from *finite-horizon* problems or problems with time-dependent dynamics and running cost. Static HJB PDEs usually result from *exit-time* or *infinite-horizon* problems with time-independent (though perhaps time-discounted) dynamics and running cost. In the time-dependent case, efficient numerical methods are typically based on time-marching. In the static case, a naive approach involves iterative solving of a system of discretized equations. Several popular approaches were developed precisely to avoid these