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FINITE ELEMENT METHODS FOR A BI-WAVE EQUATION MODELING D-WAVE SUPERCONDUCTORS*

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Abstract

In this paper we develop two conforming finite element methods for a fourth order bi-wave equation arising as a simplified Ginzburg-Landau-type model for d-wave superconductors in absence of applied magnetic field. Unlike the biharmonic operator Δ^2, the bi-wave operator $\Box^2$ is not an elliptic operator, so the energy space for the bi-wave equation is much larger than the energy space for the biharmonic equation. This then makes it possible to construct low order conforming finite elements for the bi-wave equation. However, the existence and construction of such finite elements strongly depends on the mesh. In the paper, we first characterize mesh conditions which allow and not allow construction of low order conforming finite elements for approximating the bi-wave equation. We then construct a cubic and a quartic conforming finite element. It is proved that both elements have the desired approximation properties, and give optimal order error estimates in the energy norm, suboptimal (and optimal in some cases) order error estimates in the H^1 and L^2 norm. Finally, numerical experiments are presented to guage the efficiency of the proposed finite element methods and to validate the theoretical error bounds.

Key words: Bi-wave operator, d-wave superconductors, Conforming finite elements, Error estimates.
Mathematics subject classification: 65N30, 65N12, 65N15.

1. Introduction

This paper concerns finite element approximations of the following boundary value problem:

$$\delta\Box^2 u - \Delta u = f \qquad \text{in } \Omega, \tag{1.1}$$

$$u = \partial_{\bar{n}} u = 0 \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $0 < \delta \ll 1$ is a given (small) number,

$$\Box u := \partial_{xx} u - \partial_{yy} u, \qquad \Box^2 u := \Box(\Box u),$$
$$\bar{n} := (n_1, -n_2), \qquad \partial_{\bar{n}} u := \nabla u \cdot \overline{n},$$

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$\Omega \subset \mathbf{R}^2$ is a bounded domain with piecewise smooth boundary $\partial\Omega$, and $n := (n_1, n_2)$ denotes the unit outward normal to $\partial\Omega$. As $\Box$ is the well-known (2-D) wave operator, we shall call $\Box^2$ the bi-wave operator throughout this paper. It is easy to verify that

$$\Box^2 u = \frac{\partial^4 u}{\partial x^4} - 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}.$$

Hence, equation (1.1) is a fourth order PDE, which can be viewed as a singular perturbation of the Poisson equation by the bi-wave operator. As a comparison, we recall that the biharmonic operator Δ^2 is defined as

$$\Delta^2 u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}.$$

Although there is only a sign difference in the mixed derivative term, the difference between Δ^2 and $\Box^2$ is fundamental because Δ^2 is an elliptic operator while $\Box^2$ is a hyperbolic operator.

Superconductors are materials that have no resistance to the flow of electricity when the surrounding temperature is below some critical temperature. At the superconducting state, the electrons are believed to "team up pairwise" despite the fact that the electrons have negative charges and normally repel each other. The Ginzburg-Landau theory [9] has been well accepted as a good mean field theory for low (critical temperature) T_c superconductors [11]. However, a theory to explain high T_c superconductivity still eludes modern physics. In spite of the lack of satisfactory microscopic theories and models, various generalizations of the Ginzburg-Landau-type models to account for high T_c properties such as the anisotropy and the inhomogeneity have been proposed and developed. In low T_c superconductors, electrons are thought to pair in a form in which the electrons travel together in spherical orbits, but in opposite directions. Such a form of pairing is often called s-wave [11]. However, in high T_c superconductors, experiments have produced strong evidence for d-wave pairing symmetry in which the electrons travel together in orbits resembling a four-leaf clover (cf. [4, 6, 10, 12] and the references therein). Recently, the d-wave pairing has gained substantial support over s-wave pairing as the mechanism by which high-temperature superconductivity might be explained. In generalizing the Ginzburg-Landau models to high T_c superconductors, the key idea is to introduce multiple order parameters in the Ginzburg-Landau free energy functional. These models, which can also be derived from the phenomenological Gorkov equations [6], have built a reasonable basis upon which detailed studies of the fine vortex structures in some high T_c materials have become possible. We refer the reader to [4, 6, 10, 12] and the references therein for a detailed exposition on modeling and analysis of d-wave superconductors.

We obtain equation (1.1) from the Ginzburg-Landau-type d-wave model considered in [4] (also see [10, 12]) in absence of applied magnetic field by neglecting the zeroth order nonlinear terms but retaining the leading terms. In the equation, u (notation ψ_d is instead used in the cited references) denotes the d-wave *order parameter*. We note that the original order parameter ψ_d in the Ginzburg-Landau-type model [4, 10] is a complex-valued scalar function whose magnitude represents the density of superconducting charge carriers, however, to reduce the technicalities and to present the ideas, we assume u is a real-valued scalar function in this paper and remark that the finite element methods developed in this paper can be easily extended to the complex case. We also note that the parameter δ appears in the full model as $\delta = -1/\beta$, where β is proportional to the ratio $\ln(T_{s0}/T)/\ln(T_{d0}/T)$ with T_{s0} and T_{d0} being the critical temperatures of the s-wave