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UNIFORM SUPERCONVERGENCE OF A FINITE ELEMENT METHOD WITH EDGE STABILIZATION FOR CONVECTION-DIFFUSION PROBLEMS*

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Abstract

In the present paper the edge stabilization technique is applied to a convection-diffusion problem with exponential boundary layers on the unit square, using a Shishkin mesh with bilinear finite elements in the layer regions and linear elements on the coarse part of the mesh. An error bound is proved for $\left\|\pi u - u^h\right\|_E$, where πu is some interpolant of the solution u and u^h the discrete solution. This supercloseness result implies an optimal error estimate with respect to the L_2 norm and opens the door to the application of postprocessing for improving the discrete solution.

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Key words: Convection-diffusion problems, Edge stabilization, FEM, Uniform convergence, Shishkin mesh.

1. Introduction

Consider the model convection-diffusion problem

$$-\varepsilon\Delta u - \boldsymbol{b}\cdot\nabla u + cu = f \quad \text{in } \Omega=(0,1)^2, \quad u=0 \quad \text{on } \Gamma=\partial\Omega \tag{1.1}$$

with a small perturbation parameter $0<\varepsilon\ll 1$ and $b_i \geq \beta_i > 0$ on $\bar{\Omega}$, $i=1,2$, with constants β_i. Furthermore, let

$$c + \frac{1}{2}\,\mathrm{div}\boldsymbol{b} \geq \gamma > 0 \quad \text{on } \bar{\Omega}. \tag{1.2}$$

This ensures the existence of a weak solution in $H_0^1(\Omega)\cap H^2(\Omega)$. The presence of the parameter ε causes the formation of regular layers at the outflow boundary at $x=0$ and $y=0$.

It is well known that the Galerkin finite element method applied to (1.1) with linear or bilinear finite elements leads on standard meshes to wild non-physical oscillations in the discrete solution.

The need of avoiding these oscillations and to resolve layer structures results in the use of layer-adapted meshes which are highly anisotropic and non-uniform. On the probably simplest layer-adapted mesh, the Shishkin mesh (see Section 2), standard Galerkin with $\mathcal{O}\left(N^2\right)$ mesh points was shown to converge uniformly by Stynes and O'Riordan [18] ten years ago:

$$\left|\left|\left|u-u^h\right|\right|\right|_\varepsilon \leq CN^{-1}\ln N. \tag{1.3}$$

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where here and throughout C is a constant independent of ε and $|||v|||_\varepsilon := \varepsilon^{1/2}|v|_1 + \gamma\|v\|_0$ is the ε-weighted H^1 norm. The fine mesh in the layer region induces some stability, but the computed solution still exhibits mild oscillations (see the numerical experiments in [12]). Moreover, the stiffness matrix of the generated discrete problem has eigenvalues with large imaginary parts, and consequently standard iterative methods do not solve the discrete systems efficiently. Therefore, additional stabilization seems to be necessary even when layer-adapted meshes are used.

Stynes and Tobiska [19] analyzed stabilization by the streamline-diffusion finite element method (SDFEM) on the coarse part of the Shishkin mesh. But the relatively popular SDFEM has some remarkable disadvantages: applied to systems of differential equations it causes non-physical couplings of unknowns [14], while for optimal control problems difficulties in the adjoint equation arise [2]. Consequently, recently new stabilization techniques as local projection schemes, the variational multiscale methods and edge stabilization appeared (for a survey, see [16]).

In the present paper we shall consider instead an edge stabilization technique, which is also called continuous interior penalty method (CIP). It was introduced and analyzed by Burman and Hansbo [4, 5]. On a shape-regular, locally uniform mesh they proved for linear elements

$$|||u - u^h|||_E \le C\big(\varepsilon^{1/2} + h^{1/2}\big)h|u|_2.$$

In the convection-dominated case this estimate is of little worth because $|u|_2 \to \infty$ for $\varepsilon \to 0$. However, the local estimates of [6] show that edge stabilization is fine away from the layers.

It is our aim to combine the Galerkin finite element method with bilinears on the fine part of a Shishkin mesh with edge stabilization for linear elements on the coarse part of the mesh. Numerical experiments have shown that bilinear elements should be preferred to linear elements in layer regions because they yield higher-order accuracy; see [13]. This is due to supercloseness properties of bilinear elements, cf. [10, 20]. This seems to contradict "normal" mesh design where triangles should be used at the boundary of the domain. However, in the singularly perturbed case the situation is different because of the highly anisotropic behaviour of the solution near boudaries. First rectangular boundary-layer meshes are constructed and second the remaining domain is triangulated; see also [15].

Similar to streamline-diffusion it is impossible to get a better estimate then (1.3) for the error in the energy norm. However, for the difference of the numerical solution and a certain interpolant of the exact solution we shall prove in a norm stronger than the ε-weighted norm

$$|||\pi u - u^h|||_E \le C\big(\varepsilon^{1/2}N^{-1} + N^{-3/2}\big).$$

This is a supercloseness result and allows for the application of postprocessing techniques giving a new approximate solution Pu^h whose error is significantly smaller than that of u^h.

2. Derivative Bounds and Mesh Construction

For the construction of properly adapted meshes and for the analysis of the resulting method it is essential to have precise knowledge of the behaviour of the solution and its derivatives. As in [10] we shall assume the solution of (1.1) admits the representation

$$u = S + E_1 + E_2 + E_{12}, \tag{2.1a}$$