Journal of Computational Mathematics
Vol.27, No.6, 2009, 787–801.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2009.09-m2674

SOLVING A CLASS OF INVERSE QP PROBLEMS BY A SMOOTHING NEWTON METHOD*

Xiantao Xiao and Liwei Zhang
Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
Email: xiaoxiantao82@yahoo.com.cn, lwzhang@dlut.edu.cn

Abstract

We consider an inverse quadratic programming (IQP) problem in which the parameters in the objective function of a given quadratic programming (QP) problem are adjusted as little as possible so that a known feasible solution becomes the optimal one. This problem can be formulated as a minimization problem with a positive semidefinite cone constraint and its dual (denoted IQD(A, b)) is a semismoothly differentiable (SC^1) convex programming problem with fewer variables than the original one. In this paper a smoothing Newton method is used for getting a Karush-Kuhn-Tucker point of IQD(A, b). The proposed method needs to solve only one linear system per iteration and achieves quadratic convergence. Numerical experiments are reported to show that the smoothing Newton method is effective for solving this class of inverse quadratic programming problems.

Mathematics subject classification: 90C20, 90C25, 90C90.
Key words: Fischer-Burmeister function, Smoothing Newton method, Inverse optimization, Quadratic programming, Convergence rate.

1. Introduction

For solving an optimization problem, we usually assume that the parameters, associated with decision variables in the objective function or in the constraint set, are known and we need to find an optimal solution to the problem. However, in the practice there are many instances in which we only know some estimates for parameter values, but we may have certain optimal solutions from experience, observations or experiments. An inverse optimization problem is to find values of parameters which make the known solutions optimal and which differ from the given estimates as little as possible.

Burton and Toint (1992) [3] first investigated an inverse shortest paths problem, since then there are many important contributions to inverse optimization and a large number of inverse combinatorial optimization problems have been studied, see the survey paper by Heuberger [6] and the references [1,2,4], etc. For continuous optimization, Zhang and Liu [14,15] first studied inverse linear programming, Iyengar and Kang [7] discussed inverse conic programming models and their applications in portfolio optimization. And recently, Zhang and Zhang [16] studied the rate of convergence of the augmented Lagrangian method for a type of inverse quadratic programming (IQP) problems. The quadratic programming problem, considered in [16], is of the form

$$\mathrm{QP}(G, c, A, b) \qquad \begin{aligned} \min \quad & f(x) := \frac{1}{2}x^T Gx + c^T x \\ \text{s.t.} \quad & x \in \Omega_P := \{x' \in \mathbb{R}^n \mid Ax' \geq b\}, \end{aligned} \tag{1.1}$$

* Received October 29, 2007 / Revised version received March 24, 2008 / Accepted February 20, 2009 /

where $G \in \mathbb{R}^{n\times n}$ is a symmetric matrix, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Let

$$A := (a_1, \dots, a_m)^T, \quad a_i \in \mathbb{R}^n,\, i = 1, \dots, m,$$

$\mathbb{S}^n$ denote the space of $n \times n$ symmetric matrices, and SOL(P) be the set of optimal solutions to a problem (P).

Given a feasible point $x^0 \in \Omega_P$, which should be the optimal solution to Problem (1.1) and a pair $(G^0, c^0) \in \mathcal{S}^n \times \mathbb{R}^n$ which is an estimate of (G, c). The inverse quadratic programming considered in this paper is to find a pair $(G, c) \in \mathbb{S}^n \times \mathbb{R}^n$ to solve

$$\text{IQP}(A,b) \qquad \begin{array}{ll} \min & \frac{1}{2}\|(G,c) - (G^0, c^0)\|^2 \\ \text{s.t.} & x^0 \in \text{SOL}(\text{QP}(G,c,A,b)), \\ & (G,c) \in \mathbb{S}^n_+ \times \mathbb{R}^n, \end{array} \tag{1.2}$$

where $\mathbb{S}^n_+$ is the cone of positively semi-definite symmetric matrices in $\mathbb{S}^n$ and $\|\cdot\|$ is defined by

$$\|(G', c')\| := \sqrt{\text{Tr}(G'^T G') + c'^T c'} \;\text{ for }\; (G', c') \in \mathbb{R}^{n\times n} \times \mathbb{R}^n.$$

Problem (1.2) is a cone-constrained optimization problem with a quadratic objective function. The scale of this problem will be quite large when n is a large number as the number of its decision variables is $n + n(n+1)/2$.

Without loss of generality, we assume that the first p constraints in Ω_P are active at x^0, or equivalently

$$I(x^0) := \{j : a_j^T x^0 = b_j, j = 1, \dots, m\} = \{1, \dots, p\}.$$

If $G \in \mathbb{S}^n_+$, then $x^0 \in \text{SOL}(\text{QP}(G,c,A,b))$ if and only if there exists $u \in \mathbb{R}^p$ such that

$$c + Gx^0 - \sum_{i=1}^p u_i a_i = 0,\; u_i \geq 0, i = 1, \dots, p.$$

Let $A_0 := (a_1, \dots, a_p)^T \in \mathbb{R}^{p\times n}$ and the j−th column of A_0 be $A_j \in \mathbb{R}^p$. Then $A_0 := (A_1, \dots, A_n)$ and the problem (1.2) can be equivalently expressed as follows

$$\begin{array}{ll} \min & \frac{1}{2}\|(G,c) - (G^0, c^0)\|^2 \\ \text{s.t.} & c + Gx^0 - A_0^T u = 0, \\ & (G,c,u) \in \mathbb{S}^n_+ \times \mathbb{R}^n \times \mathbb{R}^p_+. \end{array} \tag{1.3}$$

As the dimension of the above problem is $n(n+1)/2 + n + p$, quite big when n is large, it would be helpful to consider its dual. It follows from [16] that the dual problem can be written as

$$\text{IQD}(A,b) \qquad \begin{array}{ll} \max & \upsilon_0(z) \\ \text{s.t.} & A_0 z \leq 0, \end{array} \tag{1.4}$$

where

$$\upsilon_0(z) = -\frac{1}{2}\|z\|^2 + c^{0T} z - \frac{1}{2}\|\Pi_{\mathcal{S}^n_+}(\bar{G}(z))\|_F^2 + \frac{1}{2}\|G^0\|_F^2, \tag{1.5}$$

and

$$\bar{G}(z) = G^0 - \mathcal{B}z, \quad \mathcal{B}z := \frac{zx^{0T} + x^0 z^T}{2}.$$