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PARALLEL HIERARCHICAL MATRIX PRECONDITIONERS FOR THE CURL-CURL OPERATOR*

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Abstract

This paper deals with the preconditioning of the curl-curl operator. We use $\mathbf{H}(\mathbf{curl})$-conforming finite elements for the discretization of our corresponding magnetostatic model problem. Jumps in the material parameters influence the condition of the problem. We will demonstrate by theoretical estimates and numerical experiments that hierarchical matrices are well suited to construct efficient parallel preconditioners for the fast and robust iterative solution of such problems.

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1. Introduction

A major field of occurrence of the curl-curl operator is computational electromagnetism. An example is the ungauged vector potential based magnetostatic problem

$$\mathbf{curl}\frac{1}{\mu}\mathbf{curl}\,\mathbf{u} = \mathbf{j}_0 \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathbf{u_t} = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

which we choose as our model problem with given source current $\mathbf{j}_0$. For theoretical purposes we assume that $\Omega \subset \mathbb{R}^3$ is a convex (curved) polyhedron, while in pratice this property does not seem to be required. We denote $\mathbf{n}$ as the exterior normal at the boundary $\partial\Omega$ and $\mathbf{u_t} := \mathbf{u} \times \mathbf{n}$ as the tangential surface trace of the vector potential $\mathbf{u}$. A typical setting to be simulated in magnetostatics is shown in Fig. 7.1. The computational domain $\Omega = \Omega_C \cup \Omega_I$ consists of different materials that are characterized by their material parameters, i.e. their *conductivity* σ satisfying $0 \leq \sigma(x) \leq \sigma_1$ and their *magnetic permeability* $\mu := \mu_r \cdot \mu_0 \in L^\infty(\Omega)$ with $1 \leq \mu_r(x) \leq \mu_1/\mu_0$ for some constants $\mu_1, \sigma_1 \in \mathbb{R}$ and $\mu_0 := 4\pi \cdot 10^{-7}$ (Vs)/(Am). $\mathbf{j}_0$ vanishes in the non-conducting domain Ω_I.

The ungauged magnetostatic problem is singular, because any gradient field $\mathbf{grad}\,\phi$ can be added to the solution. The magnetic field $\mathbf{B} := \mathbf{curl}\,\mathbf{u}$, which is the measurable physical quantity of interest, is not affected by this alternative solution $\mathbf{u}_{\text{new}} := \mathbf{u} + \mathbf{grad}\,\phi$. The vector potential $\mathbf{u}$ itself is not a measurable physical quantity.

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The ungauged vector potential based magnetostatic problem is a special case of the vector potential based full Maxwell problem in frequency domain and temporal gauge

$$\mathbf{curl}\frac{1}{\mu}\mathbf{curl}\,\mathbf{u} + \mathrm{i}\omega(\sigma + \mathrm{i}\omega\epsilon)\mathbf{u} = -(\sigma + \mathrm{i}\omega\epsilon)\mathbf{grad}\,\varphi_0. \tag{1.2}$$

Herein, the electric permittivity is assigned by $\epsilon := \epsilon_r \cdot \epsilon_0$ with $\epsilon_r \geq 1$, $\epsilon_0 := 8.85 \cdot 10^{-12}$ (As)/(Vm) and φ_0 denotes the scalar electric potential. The magnetostatic equation (1.1a) follows from (1.2) in the case of vanishing angular frequency $\omega = 0$. The operator that arises from the full Maxwell problem is regular for all $\omega > 0$, whereas the curl-curl operator has a large kernel. Since the electromagnetic fields that emerge at low frequencies in the full Maxwell model are a good approximation of magnetostatic fields, it can be expected that the operator (1.2) at small frequencies is a good approximation for the operator (1.1). It is therefore an obvious idea to regularize the "magnetostatic operator" by adding a multiple of the identity. Hence, we consider the operator

$$\mathbf{L}_\alpha := \mathbf{curl}\frac{1}{\mu}\mathbf{curl} + \alpha\mathbf{I} \tag{1.3}$$

with constant $1/\mu_1 \leq \alpha \in \mathbb{R}$ as a preconditioner for the magnetostatic operator

$$\mathbf{L}_0 = \mathbf{curl}\frac{1}{\mu}\mathbf{curl}.$$

One of the most established methods for the iterative solution of electromagnetic problems are multigrid methods; see [1, 16]. Algebraic multigrid methods (AMG) can be applied if no finite element (FE) grid hierarchy is available; see [26] and [8] for an improved version. However, they lack a comprehensive theoretical analysis. A major difference of the method in [26] and the method presented in this article is that we do not regularize the problem itself. We rather use the regularized operator for generating preconditioners for the original problem (1.1), while in [26] an approximate solution which depends on the regularization parameter α is computed. See [18] for a preconditioning technique that relies on solvers for the discrete Poisson problem. In this article we propose the usage of hierarchical matrices ($\mathcal{H}$-matrices) [14, 15] due to their robustness with respect to non-smooth coefficients in the differential operator.

Hierarchical matrices provide a setting in which approximations of fully populated matrices (such as the inverse or the factors of the LU decomposition of sparse matrices) can be computed with logarithmic-linear complexity. The existence of such approximations in the case of FE discretizations was proved in [2, 4, 7] for general scalar elliptic boundary value problems. A strategy that is also based on $\mathcal{H}$-matrices was proposed in [27]. There, the discretization A of $\mathbf{L}_0$ is regularized by adding UU^T to A, where U is the matrix consisting of the kernel vectors of A (the so-called *discrete grad-div regularization*).

After the introduction of appropriate spaces and the variational formulation of our problem in Sect. 2, we will review hierarchical matrices in Sect. 3 in the context of nested dissection reorderings; see [10]. In Sect. 4 we will lay theoretical ground to the $\mathcal{H}$-matrix approximation of the factors of the LU decomposition in the case of discretizations of the operator (1.3) with Nédélec's edge elements [24]. The regularization (1.3) guarantees that low-precision LU factorizations can be computed in the methodology of hierarchical matrices, which can be used for preconditioning. In Sect. 5 we will investigate the influence of the regularization parameter α and the accuracy $\varepsilon_{\mathcal{H}}$ of the hierarchical matrix approximation on the condition number of the preconditioned problem. In Sect. 6 it will be shown how the nested dissection structure of the hierarchical LU decomposition can be exploited for parallelization. Finally, Sect. 7 will