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OPTIMAL ERROR ESTIMATES FOR NEDELEC EDGE ELEMENTS FOR TIME-HARMONIC MAXWELL'S EQUATIONS*

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Abstract

In this paper, we obtain optimal error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm for the Nédélec edge finite element approximation of the time-harmonic Maxwell's equations on a general Lipschitz domain discretized on quasi-uniform meshes. One key to our proof is to transform the L^2 error estimates into the L^2 estimate of a discrete divergence-free function which belongs to the edge finite element spaces, and then use the approximation of the discrete divergence-free function by the continuous divergence-free function and a duality argument for the continuous divergence-free function. For Nédélec's second type elements, we present an optimal convergence estimate which improves the best results available in the literature.

Mathematics subject classification: 65N30, 35Q60.
Key words: Edge finite element, Time-harmonic Maxwell's equations.

1. Introduction

Convergence analysis for edge element discretizations of the time-harmonic Maxwell's equations have been much studied in the literatures, see [4, 5, 7, 10, 12, 13]. Monk [12] first proved error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm under the assumption that Ω is convex, but both the exponent and the constant of convergence rate in L^2 error estimate involve an arbitrarily small constant $\varepsilon > 0$. Afterward, Hiptmair [10] and Monk [13] obtained asymptotic quasi-optimality of error estimates in $\boldsymbol{H}(\mathbf{curl})$-norm for a general Lipschitz polyhedron. Recently, Buffa [5] presented an abstract convergence theory for a class of noncoercive problems and then applied it to this model.

In this paper, we obtain optimal error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm for the Nédélec edge finite element approximation of the time-harmonic Maxwell's equations on a general Lipschitz domain and quasi-uniform meshes. First of all, we use the discrete Helmholtz decomposition for the difference between the Nédélec finite element solution and a finite element function, then obtain the discrete divergence-free function $\boldsymbol{w}_h$ which belongs to the edge finite element spaces. Secondly, we transform error estimate in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-seminorm into the L^2 estimate of $\boldsymbol{w}_h$ by proving that the error function is discrete divergence-free. Thirdly,

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we obtain the L^2 estimates of $\boldsymbol{w}_h$ by using its special approximation $\boldsymbol{w}$ which is a continuous divergence-free function and a duality argument for $\boldsymbol{w}$. Finally, we obtain optimal error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm. Compare with the results in [12], the exponent and the constant of convergence rate in our error estimates are independent of the constant ε, thereby we improve the L^2 error estimate in [12].

Combining optimal L^2 error estimates with the corresponding interpolation error estimates for Nédélec's second type elements, we obtain the convergence order of the error function, and the order only depends on the Lipschitz domain and the smoothness of the solution. Especially, for the convex domain, we obtain an optimal convergence order. It should be noted that the L^2 error estimates are one order higher than the $\boldsymbol{H}(\mathbf{curl})$-norm estimates for Nédélec's second type elements, however, it is not correct for Nédélec's first type elements, because when restricted to the elements of the triangulation they fails to provide a complete space of polynomial (see [12]).

To avoid the repeated use of generic but unspecified constants, following [18], we use the notation $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$, the above generic constants C are independent of the function under consideration, but they may depend on Ω and the shape-regularity of the meshes.

The rest of the paper is organized as follows. In Section 2, we introduce the time-harmonic Maxwell's equations, then present its corresponding equivalent variational problem and the well-posedness. In Section 3, we present the discrete variational problem and some preliminaries. In Section 4, we obtain optimal error estimates in both L^2-norm and $\boldsymbol{H}(\mathbf{curl})$-norm, and present an optimal convergence order for Nédélec's second type elements.

2. Formulation of the Problem

For simplicity, we assume that Ω is a bounded Lipschitz polyhedron in $\mathbb{R}^3$ with connected boundary Γ and unit outward normal $\boldsymbol{\nu}$. For any $m \geq 1$ and $p \geq 1$, we denote the standard Sobolev space by $W^{m,p}(\Omega)$. Especially, when $p = 2$, we denote the space by $H^m(\Omega) = W^{m,2}(\Omega)$, and $H_0^1(\Omega) = \{u \in H^1(\Omega) : \ u|_\Gamma = 0\}$. Furthermore, we also need some other Sobolev functional spaces (see [9, 14]):

$$
\begin{aligned}
\boldsymbol{H}_0(\mathbf{curl};\Omega) &= \left\{\boldsymbol{u} \in (L^2(\Omega))^3 \;\middle|\; \nabla \times \boldsymbol{u} \in (L^2(\Omega))^3,\ \boldsymbol{\nu} \times \boldsymbol{u} = \mathbf{0} \text{ on } \Gamma\right\},\\
\boldsymbol{H}^s(\mathbf{curl};\Omega) &= \left\{\boldsymbol{u} \in (H^s(\Omega))^3 \;\middle|\; \nabla \times \boldsymbol{u} \in (H^s(\Omega))^3\right\},
\end{aligned}
$$

where $s > 0$, and the above spaces are equipped with the norms, respectively,

$$
\begin{aligned}
\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} &= \left(\|\boldsymbol{v}\|_0^2 + \|\nabla \times \boldsymbol{v}\|_0^2\right)^{1/2} \quad \forall\ \boldsymbol{u} \in \boldsymbol{H}_0(\mathbf{curl};\Omega),\\
\|\boldsymbol{v}\|_{\boldsymbol{H}^s(\mathbf{curl};\Omega)} &= \left(\|\boldsymbol{v}\|_{H^s(\Omega)}^2 + \|\nabla \times \boldsymbol{v}\|_{H^s(\Omega)}^2\right)^{1/2} \quad \forall\ \boldsymbol{v} \in \boldsymbol{H}^s(\mathbf{curl};\Omega).
\end{aligned}
$$

Here, $\|\cdot\|_0$ denotes the norm in $(L^2(\Omega))^3$.

We consider the following classical time-harmonic Maxwell's equations (c.f. [10, 12, 14])

$$
\nabla \times (\mu^{-1}\nabla \times \boldsymbol{E}) - \omega^2\eta\boldsymbol{E} = \boldsymbol{F} \quad \text{in } \Omega, \tag{2.1}
$$

$$
\boldsymbol{\nu} \times \boldsymbol{E} = \mathbf{0} \quad \text{on } \Gamma, \tag{2.2}
$$

where μ is called the magnetic permeability, $\omega > 0$ is called the angular frequency, $\eta = \epsilon + i\sigma/\omega$, where $i = \sqrt{-1}$, ϵ and σ are called, respectively, the electric permittivity, and conductivity of the homogeneous isotropic body occupying Ω, $\boldsymbol{F} = i\omega\boldsymbol{J}$ with the applied current density $\boldsymbol{J}$.