Journal of Computational Mathematics, Vol.27, No.2-3, 2009, 400–424.

LOW ORDER NONCONFORMING RECTANGULAR FINITE ELEMENT METHODS FOR DARCY-STOKES PROBLEMS*

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Abstract

In this paper, we consider lower order rectangular finite element methods for the singularly perturbed Stokes problem. The model problem reduces to a linear Stokes problem when the perturbation parameter is large and degenerates to a mixed formulation of Poisson's equation as the perturbation parameter tends to zero. We propose two 2D and two 3D nonconforming rectangular finite elements, and derive robust discretization error estimates. Numerical experiments are carried out to verify the theoretical results.

Mathematics subject classification: 65N12, 65N15, 65N22, 65N30.
Key words: Darcy-Stokes problem, Finite element, Uniformly stable.

1. Introduction

We consider the following model equations on a bounded connected polygonal domain $\Omega \subset \mathbb{R}^d, d = 2, 3$. The velocity $\mathbf{u}$ and the pressure p satisfy

$$-\epsilon^2 \Delta \mathbf{u} + \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in} \quad \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = g \qquad \text{in} \quad \Omega, \tag{1.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $\epsilon \in (0, 1]$ is the perturbation parameter and the source term g is assumed to satisfy the solvability condition

$$\int_\Omega g \, d\Omega = 0. \tag{1.4}$$

Then the problem (1.1)-(1.3) admits a unique solution if the condition (1.4) is satisfied and the pressure p is determined only up to addition of a constant.

When ϵ is not too small, and $g \equiv 0$, the system (1.1)-(1.3) is simply a standard Stokes equation but with an additional nonharmful lower order term. On the other hand, when $\mathbf{f} \equiv \mathbf{0}$ and $\epsilon \to 0$, the model formally tends to a mixed formulation of Poisson's equation with

* Received December 27, 2007 / Revised version received October 10, 2008 / Accepted November 28, 2008 /

homogeneous Neumann boundary conditions. When $\epsilon = 0$ the first equation (1.1) has the form of Darcy's law for flow in a homogeneous porous medium, with $\mathbf{u}$ a volume averaged velocity.

As we know, in order to make the discrete problem for the Darcy-Stokes system well posed, one has to take some care in the choice of the velocity/pressure approximation spaces. In particular, the naive choice of piecewise linear interpolations for both the velocity and pressure or piecewise linear interpolation for the velocity and piecewise constants for the pressure will result in ill-posed discrete problem. The remedy is either to enrich the velocity space, using high order interpolation or local bubble functions, or to stabilize the method using Galerkin/least-squares formulation. A vast number of discretization and stabilization techniques for the Stokes equation have been proposed in [1,3,8,9,11,12,15,17-19,21]. For the finite element methods treating the Darcy flow we refer to [23] and the references therein. In this work, we want to treat both porous media flow and open fluid flow, so it would be advantageous if the same element could be used in both the Stokes limit and the Darcy limit [14,16,20]. A seemingly promising candidate for such an element is the nonconforming Crouzeix-Raviart (CR) element [13,24], which has several useful properties. For example, in combination with piecewise constant pressure, it satisfies the inf-sup condition and is elementwise mass conserving; it is also easy to implement. However, Mardal, Tai and Winther [22] showed that the CR element does not converge when applied to Darcy problem (or the Darcy-Stokes problem with vanishing viscosity). By adding an edge stabilization term, Burman and Hansbo [10] proved that the simplest P_1/P_0 element can be used for both Darcy and Stokes problems, but the choice of stabilization parameter requires special care.

In [22], Mardal, Tai and Winther considered the singularly perturbed Stokes problem and proposed a triangular element which behaves uniformly with respect to the perturbation parameter. Later, they generalized it to 3D cases and proposed a robust tetrahedron element in [26]. Recently, Xie, Xu and Xue [27] discussed the Darcy-Stokes interface problems and concluded that a traditional stable and uniformly-consistent Stokes element is also uniformly stable for the Darcy-Stokes-Brinkman model if and only if the pressure space contains the divergence range of the velocity space. They also developed a class of low order simplex elements for both 2D and 3D cases which are uniform, stable with respect to the viscosity coefficient, zero-order term coefficient, and their jumps.

For rectangular element cases, it seems to be difficult to construct uniformly stable lower order $H(\mathrm{div})$-element and so far there is little related work. In this paper, we follow the idea of [22,26,27] and seek for uniform stable low order rectangular elements for the Darcy-Stokes problem (1.1)-(1.3) in both 2D and 3D cases.

The rest of this paper is organized as follows. In Section 2 we discuss the general assumptions for the construction of stable finite element methods for problem (1.1)-(1.3). Based on these assumptions, two new rectangular elements in 2D cases are constructed and analyzed in Section 3. Then, in Section 4, we derive uniform discretization error estimates with respect to the perturbation parameter. In Section 5 we discuss the case allowing the viscosity to be zero in the Darcy-Stokes equation. The extension to 3D cases is considered in Section 6. Finally, some numerical experiments are given in Section 7 to verify our theoretical results.

2. General Convergence Analysis for Nonconforming Elements

Let us introduce some notations. H^k denotes the Sobolev space of scalar function whose derivatives up to order k are square integrable, with the norm $\|\cdot\|_k$. The semi-norm derived from