Journal of Computational Mathematics, Vol.27, No.2-3, 2009, 338–347.

A NOTE ON PRESSURE APPROXIMATION OF FIRST AND HIGHER ORDER PROJECTION SCHEMES FOR THE NONSTATIONARY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS*

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Abstract

Projection methods are efficient operator-splitting schemes to approximate solutions of the incompressible Navier-Stokes equations. As a major drawback, they introduce spurious layers, both in space and time. In this work, we survey convergence results for higher order projection methods, in the presence of only strong solutions of the limiting problem; in particular, we highlight concomitant difficulties in the construction process of accurate higher order schemes, such as limited regularities of the limiting solution, and a lack of accurate initial data for the pressure. Computational experiments are included to compare the presented schemes, and illustrate the difficulties mentioned.

Mathematics subject classification: 65N22, 65F05, 65M15, 35K55, 35Q30.
Key words: Incompressible Navier-Stokes equation, Time discretization, Projection method.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, for $d = 2, 3$ be a bounded Lipschitz domain, and $T > 0$; we consider the time-dependent Navier-Stokes equations for incompressible, viscous ($\nu > 0$) Newtonian fluids,

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega_T := (0,T)\times\Omega, \tag{1.1}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega_T, \tag{1.2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \partial\Omega_T := (0,T)\times\partial\Omega, \tag{1.3}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \quad \text{in } \Omega. \tag{1.4}$$

Here, $\mathbf{u} : \Omega_T \to \mathbb{R}^d$ denotes the velocity field, $p : \Omega_T \to \mathbb{R}$ the scalar pressure of vanishing mean value, i.e., $\int_\Omega p(\cdot,\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$, and a given force $\mathbf{f} : \Omega_T \to \mathbb{R}^d$ is driving the fluid flow, with initial velocity field $\mathbf{u}_0 : \Omega \to \mathbb{R}^d$.

In the following, we approximate strong solutions $\mathbf{u} \in W^{1,2}\big(0,T;\mathbf{J}_0(\Omega)\big) \cap L^2\big(0,T;\mathbf{J}_1(\Omega)\cap \mathbf{W}^{2,2}(\Omega)\big)$ of (1.1)-(1.4), whose existence for data

$$\mathbf{u}_0 \in \mathbf{J}_1(\Omega), \quad \mathbf{f} \in L^2\big(0,T;\mathbf{J}_0(\Omega)\big)$$

is well-known to be (at least) local ($d = 3$) resp. global ($d = 2$). Here and below, we adopt the standard notation of Sobolev and Bochner spaces, and use the notation

$$\mathbf{J}_0(\Omega) = \big\{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{div}\mathbf{v} = 0 \ \text{ weakly in } \Omega,\ \langle \mathbf{v},\mathbf{n}\rangle = 0 \text{ on } \partial\Omega\big\},$$
$$\mathbf{J}_1(\Omega) = \big\{\mathbf{v} \in \mathbf{W}_0^{1,2}(\Omega) : \operatorname{div}\mathbf{v} = 0 \ \text{ weakly in } \Omega\big\},$$

* Received December 12, 2007 / Revised version received June 12, 2008 / Accepted July 14, 2008 /

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^d$, and $\mathbf{n}(\mathbf{x}) \in \mathbb{S}^{d-1}$ is the unit vector field pointing outside Ω.

Recall that solutions of (1.1)-(1.4) suffer a breakdown of regularity for $t \to 0$, even in the case of smooth (initial) data. Global regularity would require data $\mathbf{u}_0$ and $\mathbf{f}$ to satisfy a nonlocal compatibility condition which is virtually uncheckable in actual cases: it is proved in [5] that global regularity may only be valid if there exists a solution $p_0 \in W^{1,2}(\Omega) \cap L^2_0(\Omega)$ of the overdetermined Neumann problem

$$\begin{aligned} \Delta p_0 &= \operatorname{div}\bigl(\mathbf{f}(0,\cdot) - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0\bigr) && \text{in } \Omega, \\ \nabla p_0 &= \nu \Delta \mathbf{u}_0 + \mathbf{f}(0,\cdot) - (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 && \text{on } \partial\Omega. \end{aligned}$$

In order to justify existence of (local) strong solutions, we always suppose that the data in (1.1)-(1.4) satisfy

(A1) (regularity of domain) The unique solution $\mathbf{w} \in \mathbf{J}_1(\Omega)$ of the stationary, incompressible Stokes problem

$$-\nu \Delta \mathbf{w} + \nabla \pi = \mathbf{g} \quad \text{in } \Omega \subset \mathbb{R}^d$$

is already in $\mathbf{J}_1(\Omega) \cap \mathbf{W}^{2,2}(\Omega)$, provided $\mathbf{g} \in \mathbf{L}^2(\Omega)$, and satisfies

$$\|\mathbf{w}\|_{\mathbf{W}^{2,2}} \leq C \, \|\mathbf{g}\|_{\mathbf{L}^2}.$$

(A2) (regularity of data) For any $T > 0$, let

$$\mathbf{u}_0 \in \mathbf{J}_1(\Omega) \cap \mathbf{W}^{2,2}(\Omega),$$

$$\mathbf{f} \in W^{2,\infty}\bigl(0,T;\mathbf{L}^2(\Omega)\bigr).$$

A second-order temporal discretization of (1.1)-(1.4) uses the Crank-Nicholson method; in [6], it has been shown that iterates $\{\bigl(\mathbf{u}^m, p^m\bigr)\}_{m>0}$ satisfy

$$\max_{1 \leq m \leq M} \tau_m \Big[\|\mathbf{u}(t_m,\cdot) - \mathbf{u}^m\|_{\mathbf{L}^2} + \sqrt{\tau_m} k \, \|p(t_m,\cdot) - p^{m-1/2}\|_{L^2} \Big] \leq C k^2, \tag{1.5}$$

where

$$\tau_m := \min\{1, t_m\}, \quad p^{m-1/2} = \frac{1}{2}\bigl\{p^m + p^{m-1}\bigr\}.$$

The practical disadvantage of implicit discretization strategies of (1.1)-(1.4) is the significant computational effort implied from the necessity to solve coupled nonlinear algebraic problems to determine (Galerkin approximations of) $(\, \mathbf{u}^m, p^m \,)$ at every time-step given by $1 \leq m \leq M$. As a consequence, splitting algorithms were developed to reduce complexity of actual computations; among them, and one of the first, is Chorin's projection method [1, 2, 13], where iterates for velocity field and pressure are independently obtained at every time-step. However, it is known that the quality of pressure iterates is deteriorated by unphysical boundary layers [3, 10]. One strategy to improve their quality is to either construct (formally) first-order schemes which are exempted from this deficiency (i.e., the Chorin-Uzawa scheme [7, Section 8], or Chorin-Penalty scheme [9]), whereas another one would be to construct higher order projection schemes, where possible boundary layers are less pronounced (i.e., the Van Kan scheme [14]). The Van Kan