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THE ELLIPSOID ARTIFICIAL BOUNDARY METHOD FOR THREE-DIMENSIONAL UNBOUNDED DOMAINS*

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Abstract

The artificial boundary method is applied to solve three-dimensional exterior problems. Two kind of rotating ellipsoids are chosen as the artificial boundaries and the exact artificial boundary conditions are derived explicitly in terms of an infinite series. Then the well-posedness of the coupled variational problem is obtained. It is found that error estimates derived depend on the mesh size, truncation term and the location of the artificial boundary. Three numerical examples are presented to demonstrate the effectiveness and accuracy of the proposed method.

Mathematics subject classification: 65N38, 65N30.
Key words: Artificial boundary method, Exterior harmonic problem, Finite element method, Natural boundary reduction, Oblate ellipsoid, Prolate ellipsoid.

1. Introduction

Many problems in science and engineering lead to solving boundary value problems of partial differential equations in unbounded domains. The main difficulty in finding the numerical solutions of these problems is the unboundedness of the domain. In the 1970s, attempts have been made to apply the finite element method (FEM) [3] and the finite difference method (FDM) [11] to solve these problems numerically. However, the standard FEM and FDM are not effective in solving these problems. Later, Brebbia, Hsiao and Wendland developed the boundary element method (BEM), which can reduce the dimension of the computational domain and is suitable for solving problems in the unbounded domains. Then Zienkiewicz and Kelly [34], Brezzi and Johnson [6], Johnson and Nedelec [25], Han [18], Costabel and Stephan [35] suggested the coupling of FEM and BEM, which allows to combine the advantages of BEM in treating linear problems over unbounded domains with those of FEM in treating linear and nonlinear problems over the complicated bounded domains.

The artificial boundary method (including the coupling of FEM and BEM) [10, 12, 14, 15, 17, 26, 30] reduces the original problem in an unbounded domain to am equivalent problem in a bounded domain with some suitable boundary condition on the artificial boundary. The standard procedure of the method is described as follows. First, an artificial boundary Γ is introduced to divide the original (unbounded) domain $\Omega^c = \mathbb{R}^3 \backslash (\Omega \cup \partial\Omega)$ into two subregions, a bounded inner region and an unbounded outer one. Next, certain boundary condition must be

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imposed on it. Lastly, the original problem is reduced to an equivalent one in the bounded region which is solved numerically. It is very important that how to design the suitable boundary condition on the artificial boundary and how to solve the coupled system. When the imposed boundary condition is exact and non-reflective and usually is expressed by the series in generalized sense, the method has been also called the DtN method by Keller [26,27] or the coupling of FEM and natural boundary element method (NBEM) by Yu [30].

Natural boundary reduction proposed by Feng and Yu [13] has advantages over the usual boundary reduction methods: the coupled bilinear form preserves automatically the symmetry and coerciveness of the original bilinear form, so not only the analysis of the discrete problem is simplified but also the optimal error estimates and the numerical stability are restored. These advantages make the coupling of FEM and natural boundary reduction natural and direct. Moreover, this coupling of FEM and NBEM [26,30,32] also permits us to combine the advantages of BEM for treating linear problems over unbounded domains and some problems with singularity with those of FEM. The coupled method was first applied to solve the elliptic problems in two-dimensional domain [32]. Later Du and Yu [8,9], Wu and Yu [27,29], Hu and Yu [21], Liu [28], Gatica [36] use the method to handle evolution equations, the problems over three-dimensional domains, nonlinear interface problems, the electromagnetic problems and nonlinear problem in incompressible elasticity, respectively.

For three-dimensional exterior problems, a spherical surface [27,29] is usually selected as the artificial boundary. However, for an cigar-shaped or flying saucer-shaped obstacles, a rotating ellipsoid boundary [22–24] is used as the artificial boundary. This turns out to be very efficient since it leads to a smaller computational domain, as shown in Fig. 1.1 and does not increase the computational complexity of the stiff matrix from boundary reduction using the rotating ellipsoid artificial boundary. On the other hand, an anisotropic exterior problem with the constant coefficients with the spherical artificial boundary can be reduced to an isotropic problem with the ellipsoid artificial boundary.

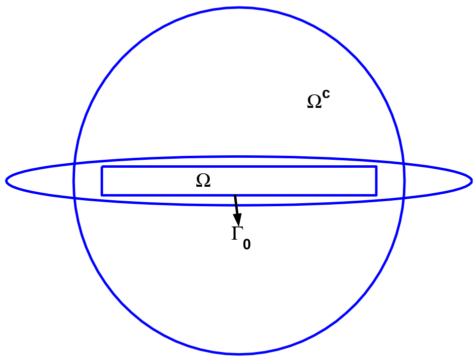

Fig. 1.1. Cross-section of cigar-shaped, ellipsoid and sphere.

Section 2 of this paper introduces two kind of rotating ellipsoids as the artificial boundaries and derives the exact artificial boundary conditions, which are expressed explicitly by the series. In Section 3, an equivalent coupled variational problem is given and the well-posedness of its continuous and discrete variational problem is obtained. In Section 4, error estimates which depend not only on the mesh size, but also on the term after truncating the series and the location of the artificial boundary [20,31] are discussed. Lastly, three numerical examples are presented to demonstrate the effectiveness and accuracy of the proposed method.