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ERROR REDUCTION IN ADAPTIVE FINITE ELEMENT APPROXIMATIONS OF ELLIPTIC OBSTACLE PROBLEMS*

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Abstract

We consider an adaptive finite element method (AFEM) for obstacle problems associated with linear second order elliptic boundary value problems and prove a reduction in the energy norm of the discretization error which leads to R-linear convergence. This result is shown to hold up to a consistency error due to the extension of the discrete multipliers (point functionals) to H^{-1} and a possible mismatch between the continuous and discrete coincidence and noncoincidence sets. The AFEM is based on a residual-type error estimator consisting of element and edge residuals. The a posteriori error analysis reveals that the significant difference to the unconstrained case lies in the fact that these residuals only have to be taken into account within the discrete noncoincidence set. The proof of the error reduction property uses the reliability and the discrete local efficiency of the estimator as well as a perturbed Galerkin orthogonality. Numerical results are given illustrating the performance of the AFEM.

Mathematics subject classification: 65N30, 65N50.
Key words: Adaptive finite element methods, Elliptic obstacle problems, Convergence analysis.

1. Introduction

Adaptive finite element methods (AFEMs) for partial differential equations based on residual- or hierarchical-type estimators, local averaging techniques, the goal-oriented dual weighted approach, or the theory of functional-type error majorants have been intensively studied during the past decades (see, e.g., the monographs [1,3,4,16,25,33] and the references therein). As far as elliptic obstacle problems are concerned, we refer to [2,5,7,8,14,19,23,26,27,31].

More recently, substantial efforts have been devoted to a rigorous convergence analysis of AFEMs, initiated in [15] for standard conforming finite element approximations of linear elliptic boundary value problems and further investigated in [24]. Using techniques from approximation theory, under mild regularity assumptions optimal order of convergence has been established in [6,29]. Nonstandard finite element methods such as mixed methods, nonconforming elements and edge elements have been addressed in [11–13]. A nonlinear elliptic boundary value problem, namely for the p-Laplacian, has been treated in [32]. The basic ingredients of the convergence

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proofs are the reliability of the estimator, its discrete local efficiency, and a bulk criterion realizing an appropriate selection of edges and elements for refinement.

For elliptic obstacle problems, the issue of error reduction in the energy functional associated with the formulation of the obstacle problem as a constrained convex minimization problem has been studied in [9] and [28]. The approach in [28] relies on techniques from nonlinear optimization, whereas the convergence analysis in [9] is restricted to the case of affine obstacles.

In this paper, we focus on the error reduction property with respect to the energy norm for general obstacles. The error estimator is of residual type and consists of element and edge residuals. The a posteriori error analysis reveals that in contrast to the unconstrained case the local residuals only have to be taken into account for elements and edges within the discrete noncoincidence set.

The paper is organized as follows: In Section 2, we introduce the elliptic obstacle problem as a variational inequality involving a closed, convex subset $K \subset H_0^1(\Omega)$ and address its unconstrained formulation in terms of a Lagrange multiplier in $H^{-1}(\Omega)$. We further consider a finite element approximation by means of P1 conforming finite elements with respect to a simplicial triangulation of the computational domain. The unconstrained formulation of the discrete approximation gives rise to discrete multipliers which are Radon measures, namely a linear combination of point functionals associated with nodal points within the discrete coincidence set. The evaluation of the discrete multipliers for the nodal basis functions of the underlying finite element space and the specification of a consistency error due to the extension of the discrete multipliers to $H^{-1}(\Omega)$ and the mismatch between the continuous and discrete coincidence and noncoincidence sets are the essential keys for the subsequent a posteriori error analysis. In Section 3, we present the error estimator, data oscillations, a bulk criterion taking care of the selection of elements and edges for refinement, and the refinement strategy. Furthermore, the main convergence result is stated in terms of a reduction of the discretization error in the energy norm up to the consistency error. The subsequent Section 4 is devoted to the proof of the error reduction property which uses the reliability of the estimator, its discrete local efficiency, and a perturbed Galerkin orthogonality as basic tools. Finally, Section 6 contains a detailed documentation of numerical results for some selected test examples displaying the convergence history of the AFEM and thus illustrating its numerical performance.

2. The Obstacle Problem and Its Finite Element Approximation

We assume $\Omega \subset \mathbb{R}^2$ to be a bounded, polygonal domain with boundary $\Gamma := \partial\Omega$. We use standard notation from Lebesgue and Sobolev space theory, refer to $H^k(\Omega)$, $k \in \mathbb{N}$, as the Sobolev spaces based on $L^2(\Omega)$, and denote their norms as $\|\cdot\|_{k,\Omega}$. We refer to $(\cdot,\cdot)_{0,\Omega}$ as the inner product of the Hilbert space $L^2(\Omega)$. For $k = 1$, $|\cdot|_{1,\Omega}$ stands for the associated seminorm on $H^1(\Omega)$ which actually is a norm on $V := H_0^1(\Omega) := \{v \in H^1(\Omega) \mid v|_\Gamma = 0\}$. We refer to $V^* := H^{-1}(\Omega)$ as the dual of V and to $\langle\cdot,\cdot\rangle$ as the associated dual pairing. Likewise, $\langle\cdot,\cdot\rangle_\Gamma$ stands for the dual pairing between the trace space $H^{1/2}(\Gamma)$ and its dual. We denote by $V_+ := \{v \in V \mid v \geq 0 \text{ a.e. on } \Omega\}$ the positive cone of V and by V_+^* the positive cone of V^*, i.e., $\sigma \in V_+^*$ iff $\langle\sigma, v\rangle \geq 0$ for all $v \in V_+$.

We further refer to $C(\overline{\Omega})$ as the Banach space of continuous functions on $\overline{\Omega}$. Its dual $\mathcal{M}(\Omega) = C(\overline{\Omega})^*$ is the space of Radon measures on Ω with $\langle\langle\cdot,\cdot\rangle\rangle$ standing for the associated dual pairing. We refer to $C_+(\overline{\Omega})$ and $\mathcal{M}_+(\Omega)$ as the positive cones of $C(\overline{\Omega})$ and $\mathcal{M}(\Omega)$. In particular, $\sigma \in \mathcal{M}_+(\Omega)$ iff $\langle\langle\sigma, v\rangle\rangle \geq 0$ for all $v \in C_+(\overline{\Omega})$.