Journal of Computational Mathematics, Vol.26, No.6, 2008, 825–837.

ON SPECTRAL METHODS FOR VOLTERRA INTEGRAL EQUATIONS AND THE CONVERGENCE ANALYSIS*

Tao Tang
Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong, China
Email: ttang@math.hkbu.edu.hk
Xiang Xu and Jin Cheng
School of Mathematics, Fudan University, Shanghai 200433, China
Email: xuxiang@fudan.edu.cn, jcheng@fudan.edu.cn

Abstract

The main purpose of this work is to provide a novel numerical approach for the Volterra integral equations based on a spectral approach. A Legendre-collocation method is proposed to solve the Volterra integral equations of the second kind. We provide a rigorous error analysis for the proposed method, which indicates that the numerical errors decay exponentially provided that the kernel function and the source function are sufficiently smooth. Numerical results confirm the theoretical prediction of the exponential rate of convergence. The result in this work seems to be the first successful spectral approach (with theoretical justification) for the Volterra type equations.

Mathematics subject classification: 35Q99, 35R35, 65M12, 65M70.
Key words: Legendre-spectral method, Second kind Volterra integral equations, Convergence analysis.

1. Introduction

This paper is concerned with the second kind Volterra integral equations

$$y(t) + \int_0^t R(t,s)y(s)ds = f(t), \qquad t \in [0,T], \tag{1.1}$$

where the source function f and the kernel function R are given, and $y(t)$ is the unknown function.

For ease of analysis, we will transfer the problem (1.1) to an equivalent problem defined in $[-1,1]$. More specifically, we use the change of variables

$$t = T(1+x)/2, \qquad x = 2t/T - 1,$$

to rewrite the Volterra equation (1.1) as follows

$$u(x) + \int_0^{T(1+x)/2} R\left(\frac{T}{2}(1+x), s\right) y(s)ds = g(x), \tag{1.2}$$

where $x \in [-1,1]$, and

$$u(x) = y\left(\frac{T}{2}(1+x)\right), \qquad g(x) = f\left(\frac{T}{2}(1+x)\right). \tag{1.3}$$

* Received March 26, 2007 / Revised version received January 24, 2008 / Accepted March 21, 2008 /

Furthermore, to transfer the integral interval $[0, T(1+x)/2]$ to the interval $[-1, x]$, we make a linear transformation: $s = T(1+\tau)/2$, $\tau \in [-1, x]$. Then, Eq. (1.2) becomes

$$u(x) + \int_{-1}^{x} K(x,\tau)u(\tau)d\tau = g(x), \qquad x \in [-1,1], \tag{1.4}$$

where

$$K(x,\tau) = \frac{T}{2}R\left(\frac{T}{2}(1+x), \frac{T}{2}(1+\tau)\right). \tag{1.5}$$

We will consider the case that the solutions of (1.1) (or equivalently (1.4)) are sufficiently smooth — in this case it is necessary to consider very high-order numerical methods such as spectral methods for approximating the solutions. There are many existing numerical methods for solving the Volterra equation (1.1), such as collocation methods, product integration methods, see, e.g., Brunner [1] and references therein. However, very few works touched the spectral approximations to (1.1). In [6], Chebyshev spectral methods are developed to solve nonlinear Volterra-Hammerstein integral equations, and in [7], Chebyshev spectral methods are investigated for Fredholm integral equations of the first kind under multiple-precision arithmetic. However, no theoretical analysis is provided to justify the high accuracy obtained. In [13], a spectral method is developed for solving (1.1), but unfortunately spectral accuracy is not observed for most of computations.

It is known that the Fredholm type equations behave more or less like a boundary value problem (see, e.g., [5]). As a result, some efficient numerical methods useful for boundary values problems (such as spectral methods) can be used directly to handle the Fredholm type equations (again see [5]). However, the Volterra equation (1.1) behaves like an initial value problem. Therefore, it is unpopular to apply the spectral approximations to the Volterra type equations. The main reason is that (1.1) is a local equation while the spectral methods use global basis functions. One of the main difficulties is how to implement the method so that spectral accuracy can be eventually obtained. On the other hand, the numerical methods for (1.1) may be different with those for the standard initial values problems in the sense that the former requires storage of all values at grid points while the latter only requires information at a fixed number of previous grid points. The storage requirement for (1.1) also makes the use of the global basis functions of the spectral methods more acceptable.

The main purpose of this work is to provide a novel numerical approach for the Volterra integral equations based on a spectral approach. We will provide a rigorous error analysis which theoretically justifies the spectral rate of convergence. This paper is organized as follows. In Section 2, we introduce the spectral approaches for the Volterra integral equations of second kind. Some lemmas useful for the convergence analysis will be provided in Section 3, and the convergence analysis, in both l^∞ and l^2 spaces, will be given in Section 4. Numerical experiments are carried out in Section 5 to verify the theoretical results obtained in Section 4.

2. Legendre-collocation Method

As demonstrated in the last section, we can assume that the solution domain is $[-1,1]$. The second kind linear integral equations in one-dimension is of the form (1.4), namely,

$$u(x) + \int_{-1}^{x} K(x,s)u(s)ds = g(x), \qquad x \in [-1,1]. \tag{2.1}$$