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# A POSTERIORI ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS OF THE CAHN-HILLIARD EQUATION AND THE HELE-SHAW FLOW*

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**Abstract**

This paper develops a posteriori error estimates of residual type for conforming and mixed finite element approximations of the fourth order Cahn-Hilliard equation $u_t + \Delta\big(\varepsilon\Delta u - \varepsilon^{-1} f(u)\big) = 0$. It is shown that the *a posteriori* error bounds depends on $\varepsilon^{-1}$ only in some low polynomial order, instead of exponential order. Using these a posteriori error estimates, we construct an adaptive algorithm for computing the solution of the Cahn-Hilliard equation and its sharp interface limit, the Hele-Shaw flow. Numerical experiments are presented to show the robustness and effectiveness of the new error estimators and the proposed adaptive algorithm.

*Mathematics subject classification:* 65M60, 65M12, 65M15, 53A10.
*Key words:* Cahn-Hilliard equation, Hele-Shaw flow, Phase transition, Conforming elements, Mixed finite element methods, A posteriori error estimates, Adaptivity.

## 1. Introduction

In this paper we derive a posteriori error estimates and develop an adaptive algorithm based on the error estimates for conforming and mixed finite element approximations of the following Cahn-Hilliard equation and its sharp interface limit known as the Hele-Shaw flow [2, 35]

$$u_t + \Delta\big(\varepsilon\Delta u - \frac{1}{\varepsilon} f(u)\big) = 0 \quad \text{in } \Omega_T := \Omega \times (0,T), \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \frac{\partial}{\partial n}\big(\varepsilon\Delta u - \frac{1}{\varepsilon} f(u)\big) = 0 \quad \text{in } \partial\Omega_T := \partial\Omega \times (0,T), \tag{1.2}$$

$$u = u_0 \quad \text{in } \Omega \times \{0\}, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^N$ $(N = 2, 3)$ is a bounded domain with $C^2$ boundary $\partial\Omega$ or a convex polygonal domain, $T > 0$ is a fixed constant, and $f$ is the derivative of a smooth double equal well potential taking its global minimum value 0 at $u = \pm 1$. In this paper we will consider the following well-known quartic potential:

$$f(u) := F'(u) \quad \text{and} \quad F(u) = \frac{1}{4}(u^2 - 1)^2.$$

For the notation brevity, we shall suppress the super-index $\varepsilon$ on $u^\varepsilon$ throughout this paper except in Section 5.

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The Eq.(1.1) was originally introduced by Cahn and Hilliard [11] to describe the complicated phase separation and coarsening phenomena in a melted alloy that is quenched to a temperature at which only two different concentration phases can exist stably. The Cahn-Hilliard equation has been widely accepted as a good (conservative) model to describe the phase separation and coarsening phenomena in a melted alloy. The function $u$ represents the concentration of one of the two metallic components of the alloy. The parameter $\varepsilon$ is an "interaction length", which is small compared to the characteristic dimensions on the laboratory scale. The Cahn-Hilliard equation (1.1) is a special case of a more complicated phase field model for solidification of a pure material [10, 27, 31]. For the physical background, derivation, and discussion of the Cahn-Hilliard equation and related equations, we refer to [2, 4, 7, 11, 13, 20, 33, 34] and the references therein. It should be noted that the Cahn-Hilliard equation (1.1) can also be regarded as the $H^{-1}$-gradient flow for the energy functional [26]

$$\mathcal{J}_\varepsilon(u) := \int_\Omega \Big[ \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2} F(u) \Big]\, dx. \tag{1.4}$$

In addition to its application in phase transition, the Cahn-Hilliard equation (1.1) has also been extensively studied in the past due to its connection to the following free boundary problem, known as the Hele-Shaw problem and the Mullins-Sekerka problem

$$\begin{aligned}
\Delta w &= 0 && \text{in } \Omega \setminus \Gamma_t,\ t \in [0,T]\,, && (1.5)\\
\frac{\partial w}{\partial n} &= 0 && \text{on } \partial\Omega,\ t \in [0,T]\,, && (1.6)\\
w &= \sigma\kappa && \text{on } \Gamma_t,\ t \in [0,T]\,, && (1.7)\\
V &= \frac{1}{2}\Big[\frac{\partial w}{\partial n}\Big]_{\Gamma_t} && \text{on } \Gamma_t,\ t \in [0,T]\,, && (1.8)\\
\Gamma_0 &= \Gamma_{00} && \text{when } t = 0\,. && (1.9)
\end{aligned}$$

Here

$$\sigma = \int_{-1}^{1} \sqrt{\frac{F(s)}{2}}\, \mathrm{d}s\,.$$

$\kappa$ and $V$ are, respectively, the mean curvature and the normal velocity of the interface $\Gamma_t$, $n$ is the unit outward normal to either $\partial\Omega$ or $\Gamma_t$,

$$[\frac{\partial w}{\partial n}]_{\Gamma_t} := \frac{\partial w^+}{\partial n} - \frac{\partial w^-}{\partial n},$$

and $w^+$ and $w^-$ are respectively the restriction of $w$ in $\Omega_t^+$ and $\Omega_t^-$, the exterior and interior of $\Gamma_t$ in $\Omega$.

Under certain assumption on the initial datum $u_0$, it was first formally proved by Pego [35] that, as $\varepsilon \searrow 0$, the function

$$w^\varepsilon := -\varepsilon\Delta u^\varepsilon + \varepsilon^{-1} f(u^\varepsilon),$$

known as the chemical potential, tends to $w$, which, together with a free boundary $\Gamma := \cup_{0\le t\le T}(\Gamma_t \times \{t\})$ solves (1.5)-(1.9). Also $u^\varepsilon \to \pm 1$ in $\Omega_t^\pm$ for all $t \in [0,T]$, as $\varepsilon \searrow 0$. The rigorous justification of this limit was carried out by Alikakos, Bates and Chen in [2] under the assumption that the above Hele-Shaw (Mullins-Sekerka) problem has a classical solution. Later, Chen [13] formulated a weak solution to the Hele-Shaw (Mullins-Sekerka) problem and showed, using an energy method, that the solution of (1.1)-(1.3) approaches, as $\varepsilon \searrow 0$, a weak solution of the Hele-Shaw (Mullins-Sekerka) problem. One of the consequences of the connection