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A TAILORED FINITE POINT METHOD FOR THE HELMHOLTZ EQUATION WITH HIGH WAVE NUMBERS IN HETEROGENEOUS MEDIUM*

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Abstract

In this paper, we propose a tailored-finite-point method for the numerical simulation of the Helmholtz equation with high wave numbers in heterogeneous medium. Our finite point method has been tailored to some particular properties of the problem, which allows us to obtain approximate solutions with the same behaviors as that of the exact solution very naturally. Especially, when the coefficients are piecewise constant, we can get the exact solution with only one point in each subdomain. Our finite-point method has uniformly convergent rate with respect to wave number k in L^2-norm.

Mathematics subject classification: 65N99, 74J05, 74J40.
Key words: Tailored finite point method, Helmholtz equation, Inhomogeneous media, High frequency wave.

1. Introduction

We are interested in the inhomogeneous Helmholtz equation in one-dimensional case:

$$\frac{d}{dx}\left(c^2(x)\frac{du}{dx}\right) + k^2n^2(x)u = f(x), \quad \forall x \in \Omega = (a,b) \subset \mathbb{R}, \tag{1.1}$$

$$u(a) = 0, \quad (cu' - \mathrm{i}knu)(b) = 0, \tag{1.2}$$

$$u(x) \quad \text{and} \quad c^2(x)u'(x) \quad \text{are continuous on } \Omega, \tag{1.3}$$

where 'i' is the imaginary unit, $k > 0$, $f \in L^2(\Omega)$, $c(x)$ and $n(x)$ are two piecewise smooth functions which represent the local speed of sound and the index of refraction respectively and satisfy

$$0 < c_0 \le c(x) \le C_0 < \infty, \qquad 0 < n_0 \le n(x) \le N_0 < \infty. \tag{1.4}$$

The above boundary value problem of the Helmholtz equation arises in many physical fields, such as the acoustic wave propagation, the electromagnetic wave propagation, seismic wave propagation in geophysics, and so on. It is well known that the numerical simulation of the Helmholtz equation with high wave numbers in inhomogeneous medium is extremely difficult, see, e.g., [2, 13, 14, 15]. In the last ten years, there have been some efficient methods for this kind of problems with constant coefficients, including the discrete singular convolution method [4], hybrid numerical asymptotic method [10], spectral approximation method [20], element-free Galerkin method [21, 23], the so-called ultra weak variational formulation [12], etc. Generally speaking, one needs the restriction $kh = \mathcal{O}(1)$ for the mesh size h to achieve a satisfactory numerical result.

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For our model problem (1.1)-(1.3), if we let

$$y(x) = \int_a^x \frac{1}{c^2(\xi)} d\xi, \quad \tau = y(b), \quad m(y) \equiv c\big(x(y)\big)n\big(x(y)\big), \quad F(y) \equiv f\big(x(y)\big), \tag{1.5}$$

then the function $U(y) \equiv u\big(x(y)\big)$ will satisfy the following equivalent problem:

$$U''(y) + k^2 m^2(y)U(y) = F(y), \quad y \in I = (0, \tau), \tag{1.6}$$

$$U(0) = 0, \qquad U'(\tau) - \mathrm{i}km(\tau)U(\tau) = 0, \tag{1.7}$$

$$U(y) \quad \text{and} \quad U'(y) \quad \text{are continuous on } I. \tag{1.8}$$

In this paper, we propose an approach which is based on the properties of the localized approximate problem to solve our model problem (1.6)-(1.8). Our method can give a natural approximation of the original problem with its essential properties. In particular, we can give the *exact solution* when m is a piecewise-constant function.

The rest part of this paper is organized as follows. In Section 2, we fix notations and discuss the stability results for our model problem. In Section 3, we present our finite-point method for the inhomogeneous Helmholtz equation based on the properties of the solutions. We also give the stability analysis and error estimates for the proposed method. In Section 4, some numerical examples are given to show the efficiency of our method. Finally, we make a short conclusion in Section 5.

2. Stability Analysis for Analytical Solution

Without loss of generality, we assume that $I = (0, \tau) \equiv (0, 1)$. Let

$$L^2(I) = \left\{ v \,\middle|\, \int_I |v(y)|^2 dy < +\infty \right\}$$

denote the space of all square-integrable complex-valued functions equipped with the inner product

$$(v, w) := \int_I v(y)\bar{w}(y) dy$$

and the norm

$$\|v\|_{0,I} := \sqrt{(v, v)}.$$

We also introduce the standard Sobolev spaces, for $l \in \mathbb{N}$,

$$H^l(I) = \left\{ v \,\middle|\, v \in L^2(I),\ v^{(j)} \in L^2(I),\ j = 1, \cdots, l \right\},$$

where $v^{(j)}$ are the derivatives of order j in the distribution sense. By $|v|_{l,I} := \|v^{(l)}\|_{0,I}$ a semi-norm is given in $H^l(I)$. A norm of the space $H^l(I)$ is defined as

$$\|v\|_{l,I} = \left(\sum_{j=0}^{l} |v|_{j,I}^2 \right)^{1/2}.$$

From now on, if not stated otherwise, all constants C, or C_j, with $j \in \mathbb{N}$, are assumed to be independent of all parameters of the given estimate, and having, in general, different meanings