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FINITE ELEMENT AND DISCONTINUOUS GALERKIN METHOD FOR STOCHASTIC HELMHOLTZ EQUATION IN TWO- AND THREE-DIMENSIONS*

Yanzhao Cao
Department of Mathematics, Florida A&M University, Tallahassee, FL32307, USA
Email: caoy@scs.fsu.edu
Ran Zhang and Kai Zhang
Department of Mathematics, Jilin University, Changchun 130023, China
Email: zhangran@jlu.edu.cn, kzhang@jlu.edu.cn

Abstract

In this paper, we consider the finite element method and discontinuous Galerkin method for the stochastic Helmholtz equation in $\mathbb{R}^d$ ($d = 2, 3$). Convergence analysis and error estimates are presented for the numerical solutions. The effects of the noises on the accuracy of the approximations are illustrated. Numerical experiments are carried out to verify our theoretical results.

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Key words: Stochastic partial differential equation, Finite element method, Discontinuous Galerkin method, Stochastic Helmholtz equation.

1. Introduction

Many physical and engineering phenomena are modeled by partial differential equations which often contain some levels of uncertainty. The advantage of modeling using these so-called stochastic partial differential equations (SPDEs) is that SPDEs are able to more fully capture the behavior of interesting phenomena; it also means that the corresponding numerical analysis of the model will require new tools to model the systems, produce the solutions, and analyze the information stored within the solutions. In the last decade, many researchers have studied different SPDEs and various numerical methods and approximation schemes for SPDEs have also been developed, analyzed, and tested [1, 4, 7, 8, 9, 10, 12, 13, 22]. In [4, 12], the analysis based on the traditional finite element method was successfully used on partial differential equations with random coefficients, using the tensor product between the deterministic and random variable spaces. Numerical methods for SPDEs with random forcing terms have also been studied in [7, 9].

In this paper, we study the following stochastic Helmholtz equation driven by an additive white noise forcing term:

$$\begin{cases} \Delta u(x) + k^2 u(x) = -f(x) - \sigma(x)\dot{W}(x), & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain in $\mathbb{R}^d$ ($d = 2, 3$) with smooth boundary, f is a given deterministic function and $\dot{W}$ denotes the white noise. To simplify our presentation we assume that the coefficient of the white noise is $\sigma(x) \equiv 1$. Also we assume throughout the paper that

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the wave-number k is positive and bounded away from zero, i.e., $k \geq k_0 > 0$. Following the approach of [5], the existence and uniqueness of the weak solution for (1.1) can be established by converting the problem into the Hammerstein integral equation using the Green's function. Numerical studies for Helmholtz equation have been developed for various algorithms as well and we refer to [15, 17, 21] and references therein for details about the rich literature.

The goal of this work is to construct numerical solutions of (1.1) using finite element and discontinuous Galerkin approximations and derive error estimates. As pointed out in [7], the difficulty in the error analysis of finite element methods and numerical approximations for an SPDE in general is the lack of regularity of its solution. To overcome such a difficulty, we follow the approach of [1] and [7] by first discretizing the white noise and then applying standard finite element methods and discontinuous Galerkin methods to the SPDE with discretized white noise forcing terms. To the best of our knowledge, there has been no work in the literature which studies the finite element method and discontinuous Galerkin method for the stochastic Helmholtz equation in $\mathbb{R}^d$ $(d = 2, 3)$. Here we emphasize that the discontinuous Galerkin method ([19, 20]) is particularly important for two reasons.

1. For large wave-number, the standard finite element method is inadequate for solving the Helmholtz equation, especially in the three-dimensional case, because of the pollution effect of the numerical solution.

2. To solve the stochastic Helmholtz equation using, for example, the Monte Carlo method, one needs many solves for the deterministic problem. This makes the construction of an efficient deterministic solver such as the DG method absolutely essential.

The key to the error estimates is the Lipschitz type regularity properties of the Green functions in the L^2 norm. In the two-dimensional case, such an regularity result was obtained in [7]. In the three-dimensional case, a similar result was obtained in [5]. In this paper we derive a new estimate which is sharper than the one in [5] for the regularity of the Green function. As a result we obtain error estimates for the finite element and discontinuous Galerkin approximations in the 3-D case which are comparable to finite difference error estimates (see [11]).

The paper is organized as follows. In Section 2, we study the approximation of (1.1) using discretized white noises. We shall establish the estimate of the approximate solutions in H^2-norm and their error estimates in the L^2-norm. In Section 3, we study finite element approximations and discontinuous Galerkin approximations of the stochastic Helmholtz equation with discretized white noise forcing terms, and obtain the L^2 error estimates between the finite element solutions and the exact solution of (1.1). In Section 4, we present numerical simulations using the finite element method and discontinuous Galerkin method constructed in Section 3. Finally in Section 5, we conclude the paper with a few concluding remarks.

2. The Approximation Problem

In this section, we first introduce an approximate problem of (1.1) by replacing the white noise $\dot{W}$ with its piecewise constant approximation $\dot{W}^s$. Then we establish the regularity of the solution of the approximate problem and its error estimates. For the simplicity of presentation, we assume that Ω is a convex polygonal domain.

Let $\{\mathcal{T}_h\}$ be a family of triangulations on Ω consisting of simplices. For $K \in \{\mathcal{T}_h\}$, let $h_K = \text{diam}K$ and $\rho_K =$ the radius of the largest ball inscribed in K. We say an element