Journal of Computational Mathematics, Vol.26, No.3, 2008, 284–296.

ON MAXWELL EQUATIONS WITH THE TRANSPARENT BOUNDARY CONDITION*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

In this paper we show the well-posedness and stability of the Maxwell scattering problem with the transparent boundary condition. The proof depends on the well-posedness of the time-harmonic Maxwell scattering problem with complex wave numbers which is also established.

Mathematics subject classification: 35L50, 35B35.
Key words: Electromagnetic scattering, Well-posedness, Stability.

1. Introduction

We consider the electromagnetic scattering problem with the perfect conducting boundary condition on the obstacle

$$\varepsilon\frac{\partial \mathbf{E}}{\partial t} - \nabla\times\mathbf{H} = \mathbf{J} \quad \text{in } [\mathbb{R}^3\backslash\bar{D}]\times(0,T), \tag{1.1}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} + \nabla\times\mathbf{E} = 0 \quad \text{in } [\mathbb{R}^3\backslash\bar{D}]\times(0,T), \tag{1.2}$$

$$\mathbf{n}\times\mathbf{E} = 0 \quad \text{on } \Gamma_D\times(0,T), \tag{1.3}$$

$$\mathbf{E}|_{t=0} = \mathbf{E}_0, \quad \mathbf{H}|_{t=0} = \mathbf{H}_0. \tag{1.4}$$

Here $D\subset\mathbb{R}^3$ is a bounded domain with Lipschitz boundary Γ_D, $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and $\mathbf{n}$ is the unit outer normal to Γ_D. The applied current $\mathbf{J}$ and the initial conditions $\mathbf{E}_0, \mathbf{H}_0$ are assumed to be supported in the circle $B_R = \{x\in\mathbb{R}^2 : |x| < R\}$ for some $R > 0$. The electric permittivity ε and magnetic permeability μ are assumed to be positive constants. We remark that the results in this paper can be easily extended to solve scattering problems with other boundary conditions such as the impedance boundary condition on Γ_D.

One of the fundamental problems in the efficient simulation of the wave propagation is the reduction of the exterior problem which is defined in the unbounded domain to the problem in the bounded domain. The transparent boundary condition plays an important role in the construction of various approximate absorbing boundary conditions for the simulation of wave

* Received February 21, 2008 / accepted March 3, 2008 /

propagation, see the review papers Givoli [5], Tsynkov [11], Hagstrom [7] and the references therein. The purpose of this paper is to study the transparent boundary condition for Maxwell scattering problems.

For any $s \in \mathbb{C}$ such that $\mathrm{Re}\,(s) > 0$, we let $\mathbf{E}_{\mathrm{L}} = \mathscr{L}(\mathbf{E})$ and $\mathbf{H}_{\mathrm{L}} = \mathscr{L}(\mathbf{H})$ be respectively the Laplace transform of $\mathbf{E}$ and $\mathbf{H}$ in time

$$\mathbf{E}_{\mathrm{L}}(x,s) = \int_0^\infty e^{-st}\mathbf{E}(x,t)dt, \qquad \mathbf{H}_{\mathrm{L}}(x,s) = \int_0^\infty e^{-st}\mathbf{H}(x,t)dt.$$

Since $\mathscr{L}(\partial_t \mathbf{E}) = s\mathbf{E}_{\mathrm{L}} - \mathbf{E}_0$ and $\mathscr{L}(\partial_t \mathbf{H}) = s\mathbf{H}_{\mathrm{L}} - \mathbf{H}_0$, by taking the Laplace transform of (1.1) and (1.2) we get

$$\varepsilon(s\mathbf{E}_{\mathrm{L}} - \mathbf{E}_0) - \nabla\times\mathbf{H}_{\mathrm{L}} = \mathbf{J}_{\mathrm{L}} \quad \text{in } \mathbb{R}^3\backslash\bar{D}, \tag{1.5}$$

$$\mu(s\mathbf{H}_{\mathrm{L}} - \mathbf{H}_0) + \nabla\times\mathbf{E}_{\mathrm{L}} = 0 \quad \text{in } \mathbb{R}^3\backslash\bar{D}, \tag{1.6}$$

where $\mathbf{J}_{\mathrm{L}} = \mathscr{L}(\mathbf{J})$. Because $\mathbf{J}$, $\mathbf{E}_0$, $\mathbf{H}_0$ are supported inside $B_R = \{x \in \mathbb{R}^2 : |x| < R\}$, we know that $\mathbf{E}_{\mathrm{L}}$ satisfies the time-harmonic Maxwell equation outside B_R

$$\nabla\times\nabla\times\mathbf{E} - k^2\mathbf{E}_{\mathrm{L}} = \mathbf{0} \quad \text{in } \mathbb{R}^{\mathbf{3}}\backslash\bar{\mathbf{D}},$$

where the wave number $k = \mathbf{i}\sqrt{\varepsilon\mu}s$ so that $\mathrm{Im}\,(k) = \sqrt{\varepsilon\mu}s_1 > 0$. Let $G_e : \mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_{\mathrm{R}}) \to \mathbf{H}^{-1/2}(\mathrm{Div};\Gamma_{\mathrm{R}})$ be the Dirichlet to Neumann operator

$$G_e(\hat{\mathbf{x}}\times\mathbf{E}_{\mathrm{L}}) = \frac{1}{\mathbf{i}k}\hat{\mathbf{x}}\times(\nabla\times\mathbf{E}_{\mathrm{L}}) = -\frac{1}{\sqrt{\varepsilon\mu}}\frac{1}{s}\hat{\mathbf{x}}\times(\nabla\times\mathbf{E}_{\mathrm{L}}).$$

By using (1.6) we have

$$G_e(\hat{\mathbf{x}}\times\mathbf{E}_{\mathrm{L}}) = \sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\mathbf{H}_{\mathrm{L}} \quad \text{on } \Gamma_R. \tag{1.7}$$

For $\hat{\mathbf{x}}\times\mathbf{E}_{\mathrm{L}}|_{\Gamma_R} = \sum_{n=1}^\infty\sum_{m=-n}^n a_{mn}\mathbf{U}_n^m(\hat{\mathbf{x}}) + b_{mn}\mathbf{V}_n^m(\hat{\mathbf{x}})$, we know that (cf., e.g., in Monk [9] and also the discussion in Section 2)

$$G_e(\hat{\mathbf{x}}\times\mathbf{E}_{\mathrm{L}}) = \sum_{n=1}^\infty\sum_{m=-n}^n \frac{-\mathbf{i}kRb_{mn}h_n^{(1)}(kR)}{z_n^{(1)}(kR)}\mathbf{U}_n^m + \frac{a_{mn}z_n^{(1)}(kR)}{\mathbf{i}kRh_n^{(1)}(kR)}\mathbf{V}_n^m,$$

where $\mathbf{U}_n^m, \mathbf{V}_n^m$ are the vector spherical harmonics, $h_n^{(1)}(z)$ is the spherical Hankel function of the first order of order n, and $z_n^{(1)}(z) = h_n^{(1)}(z) + zh_n^{(1)'}(z)$.

By taking the inverse Laplace transform of (1.7) we obtain the following transparent boundary condition for the electromagnetic scattering problems

$$\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{x}}\times\mathbf{H} = (\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(\hat{\mathbf{x}}\times\mathbf{E}|_{\Gamma_R}) \quad \text{on } \Gamma_R, \tag{1.8}$$

where

$$\begin{aligned}
&(\mathscr{L}^{-1}\circ G_e\circ\mathscr{L})(\hat{\mathbf{x}}\times\mathbf{E}|_{\Gamma_R})\\
&= \sum_{n=1}^\infty\sum_{m=-n}^n \left[\mathscr{L}^{-1}\left(\frac{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right) * b_{mn}(R,t)\right]\mathbf{U}_n^m\\
&\qquad - \left[\mathscr{L}^{-1}\left(\frac{z_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}{\sqrt{\varepsilon\mu}sRh_n^{(1)}(\mathbf{i}\sqrt{\varepsilon\mu}sR)}\right) * a_{mn}(R,t)\right]\mathbf{V}_n^m,
\end{aligned} \tag{1.9}$$