Journal of Computational Mathematics, Vol.25, No.5, 2007, 561–572.

CONDITION NUMBER FOR WEIGHTED LINEAR LEAST SQUARES PROBLEM *

Yimin Wei
(*School of Mathematical Sciences and Key Laboratory of Mathematics for Nonlinear Sciences, Fudan University, Shanghai, 200433, P.R. China*
Email: ymwei@fudan.edu.cn)

Huaian Diao
(*Department of Mathematics, The City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong*
Email: 50007445@student.cityu.edu.hk)

Sanzheng Qiao
(*Department of Computing and Software, McMaster University, Hamilton, Ontario, L8S 4L7, Canada*
Email: qiao@mcmaster.ca)

Abstract

In this paper, we investigate the condition numbers for the generalized matrix inversion and the rank deficient linear least squares problem: $\min_x \|Ax-b\|_2$, where A is an m-by-n ($m \geq n$) rank deficient matrix. We first derive an explicit expression for the condition number in the weighted Frobenius norm $\| [AT, \beta b] \|_F$ of the data A and b, where T is a positive diagonal matrix and β is a positive scalar. We then discuss the sensitivity of the standard 2-norm condition numbers for the generalized matrix inversion and rank deficient least squares and establish relations between the condition numbers and their condition numbers called level-2 condition numbers.

Mathematics subject classification: 15A12, 65F20.
Key words: Moore-Penrose inverse, Condition number, Linear least squares.

1. Introduction

In this paper, we consider a condition number for the linear least squares (LLS) problem [8-11]

$$\min_x \|Ax-b\|_2,$$

where $A \in \mathbf{R}^{m\times n}$ ($m \geq n$) is a rank deficient matrix. The condition number for the LLS problem with full rank is well studied, see, e.g., [3]. In the standard 2-norm analysis, the condition number is defined as

$$\mathrm{cond}(A,b) = \lim_{\epsilon\to 0^+} \sup \left\{ \frac{\|(A+E)^\dagger(b+f) - A^\dagger b\|_2}{\epsilon \, \|A^\dagger b\|_2}, \quad \|E\|_2 \leq \epsilon \|A\|_2,\ \|f\|_2 \leq \epsilon \|b\|_2 \right\},$$

where $A^\dagger$ is the Moore-Penrose inverse of A defined as the unique matrix X satisfying

$$AXA = A, \quad XAX = X, \quad (AX)^{\mathrm{T}} = AX, \quad \text{and} \quad (XA)^{\mathrm{T}} = XA,$$

where A^{T} is the transpose of A [7].

* Received December 15, 2005; final revised September 1, 2006; accepted October 1, 2006.

The condition number discussed in this section is based on a general theory of condition introduced by Rice [6]. In the context of the LLS, the problem is viewed as a mapping from a pair (A, b) to the LLS solution $x_{\text{LS}} = A^{\dagger}b$. The norm of a pair (A, b) in the domain of the mapping is defined by the weighted Frobenius norm:

$$\|[AT \quad \beta b]\|_{\text{F}}, \tag{1.1}$$

where T is positive and diagonal and $\beta > 0$. The weights T and β provide flexibility. Later, we will show that a large diagonal of T allows perturbation on b only and a large β allows perturbation on A only. The norm of a solution x in the image of the mapping is chosen as the Euclidean norm $\|x\|_2$.

In Rice's theory of condition, an absolute δ-condition is first defined by:

$$\mu_\delta = \inf\{\sigma \mid \|[ET \quad \beta f]\|_{\text{F}} \leq \delta \Rightarrow \|(A+E)^{\dagger}(b+f) - A^{\dagger}b\|_2 \leq \sigma\delta\}. \tag{1.2}$$

This definition says the image of a δ-neighborhood of a pair (A, b) is contained in a $\sigma\delta$-neighborhood of the solution $A^{\dagger}b$. So, σ is an upper bound for the magnification of the mapping and μ_δ is the least upper bound. Then, the asymptotic absolute condition number for the weighted LLS problem in the norms chosen above is

$$\mu = \lim_{\delta \to 0} \mu_\delta.$$

The relative condition number is defined by

$$\nu = \frac{\|[AT \quad \beta b]\|_{\text{F}}}{\|A^{\dagger}b\|_2}\mu.$$

As explained above, similar to the standard condition number, the δ-condition in (1.2) measures the enlargement of the mapping from (A, b) to $A^{\dagger}b$. What is different from the standard condition number is that the weighted Frobenius norm is used in the domain space of pairs (A, b).

Gratton [4] considered the case when $T = \alpha I$ $(\alpha > 0)$, and A is of full column rank and gave the expression of condition number for LLS problem.

In this paper, we consider the case when A is rank deficient under the condition that the perturbation E on A satisfies

$$\text{range}(E) \subseteq \text{range}(A) \quad \text{and} \quad \text{range}(E^{\text{T}}) \subseteq \text{range}(A^{\text{T}}), \tag{1.3}$$

where range(E) denotes the column space of E.

The rest of the paper is organized as follows. The absolute and relative condition numbers in the weighted Frobenius norm are given in Section 2. Then, in Section 3, we analyze the sensitivity of the generalized matrix inversion condition number and the rank deficient LLS condition number, called level-2 condition numbers introduced by Higham [5].

2. Condition Numbers

In this section, we present explicit expressions for the absolute and relative condition numbers for the rank deficient LLS problem in the weighted Frobenius norm described in the previous section.