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ON THE RAYLEIGH QUOTIENT FOR SINGULAR VALUES *

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Abstract

In this paper, the theoretical analysis for the Rayleigh quotient matrix is studied, some results of the Rayleigh quotient (matrix) of Hermitian matrices are extended to those for arbitrary matrix on one hand. On the other hand, some unitarily invariant norm bounds for singular values are presented for Rayleigh quotient matrices. Our results improve the existing bounds.

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1. Introduction

Let $C^{m\times n}$ be the set of all $m\times n$ complex matrices, and let $C^m = C^{m\times 1}$. Without loss of generality we always assume that $m \geq n$ in this paper. By $\|\cdot\|$ we denote a unitarily invariant norm. Especially, by $\|\cdot\|_2$ and $\|\cdot\|_F$ we denote the spectral norm and the Frobenius norm, respectively. A^* stands for the conjugate transpose of a matrix A. Let $\sigma(A)$ be the set of the singular values of A, I_k be the identity matrix of order k. For the column vectors x and y, the angle $\theta(x,y)\in[0,\frac{\pi}{2}]$ between x and y is defined by

$$\theta(x,y) = \arccos\frac{|x^*y|}{\sqrt{x^*x\cdot y^*y}}.$$

More generally, the canonical angle matrix $\Theta(X,\widetilde{X})$ between two subspaces spanned by the columns of $X\in C^{n\times k}$ and $\widetilde{X}\in C^{n\times k}$ is defined as [1]

$$\Theta(X,\widetilde{X}) = \mathrm{diag}(\theta_1,...,\theta_k),$$

where X and $\widetilde{X}$ have orthonormal columns, $\pi/2\geq\theta_1\geq...\geq\theta_k\geq 0$ and $\{\cos\theta_i\}_{i=1}^k$ are the singular values of $X^*\widetilde{X}$.

Let $A\in C^{n\times n}$ be a Hermitian matrix. The Rayleigh quotient of A with respect to $x\in C^n$ is defined by

$$\rho(x) = \frac{x^*Ax}{x^*x}, \quad 0\neq x\in C^n.$$

More generally, let $\tilde{U}_k\in C^{n\times k}$ $(1<k\leq n)$ and $\tilde{U}_k^*\tilde{U}_k = I_k$. Then the matrix

$$N = \tilde{U}_k^*A\tilde{U}_k$$

is called the Rayleigh quotient matrix of the Hermitian matrix A with respect to $\tilde{U}_k$.

The Rayleigh quotient (matrix) plays an important role in computing eigenvalues and eigenvectors. In particular, it can be applied to the eigenvector computations in Principal Component Analysis in image processing [8]. It has been studied by many authors (see, e.g., [2,4-8,14,15]).

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Theorem 1.1. [1] *Let $A \in C^{n\times n}$ be a Hermitian matrix, and let λ be the eigenvalue of A and u be the eigenvector corresponding to λ. If a vector $\tilde{u}$ satisfies* $\sin\theta(u,\tilde{u}) = \mathcal{O}(\varepsilon)$*, then*

$$\rho(\tilde{u}) = \frac{\tilde{u}^*A\tilde{u}}{\tilde{u}^*\tilde{u}} = \lambda + \mathcal{O}(\varepsilon^2).$$

Theorem 1.1 shows that the precision of the Rayleigh quotient $\rho(\tilde{u})$ as an approximate eigenvalue of a Hermitian matrix A is higher than that of $\tilde{u}$ as its approximate eigenvector. The converse of Theorem 1.1 was considered by Li [8] who obtained the following theorem.

Theorem 1.2. [8] *Let $A \in C^{n\times n}$ be a Hermitian matrix with eigenvalues*

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \tag{1.1}$$

and corresponding orthonormal eigenvectors $u_1, u_2, \cdots, u_n$. If

$$\frac{\tilde{u}_1^*A\tilde{u}_1}{\tilde{u}_1^*\tilde{u}_1} \geq \lambda_1 - \varepsilon^2,$$

where $\varepsilon \geq 0$, then

$$\sin\theta(u_1,\tilde{u}_1) \leq \frac{\varepsilon}{\sqrt{\lambda_1-\lambda_2}}.$$

Furthermore, Li [8] extended Theorem 1.2 to a more general case.

Theorem 1.3. [8] *Let $A \in C^{n\times n}$ be a Hermitian matrix with eigenvalues* (1.1) *and corresponding eigenvectors* (1.2)*. Let $U_k = (u_1, ..., u_k)$, and let $\tilde{U}_k$ be $n\times k$ and have orthonormal columns. If*

$$trace(\tilde{U}_k^*A\tilde{U}_k) \geq \lambda_1 + ... + \lambda_k - \varepsilon^2,$$

where $\varepsilon \geq 0$, then

$$||\sin\Theta(U_k,\tilde{U}_k)||_2 \leq \frac{\varepsilon}{\sqrt{\lambda_k-\lambda_{k+1}}}.$$

The following theorem provides a bound on the eigenvalues of $\tilde{U}_k^*A\tilde{U}_k$ as an approximation to those of A (see [5,11]).

Theorem 1.4. [5, 11] *Let $A \in C^{n\times n}$ be a Hermitian matrix with eigenvalues* (1.1) *and $\tilde{U}_k \in C^{n\times k}$ have orthonormal columns. Let $N = \tilde{U}_k^*A\tilde{U}_k$ and $R = A\tilde{U}_k - \tilde{U}_kN$. If the eigenvalues of N are $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_k$, then there is a permutation τ of $\{1,2,\cdots,n\}$ such that*

$$\sqrt{\sum_{i=1}^{k}(\nu_i - \lambda_{\tau(i)})^2} \leq \|R\|_F.$$

The Rayleigh quotient of Hermitian matrices for eigenvalue problems can be extended to the Rayleigh quotient (matrix) of an arbitrary matrix for singular value problems. Let

$$\mathcal{X} = \{x \mid x \in C^m, \|x\|_2 = 1\} \qquad \mathcal{Y} = \{y \mid y \in C^n, \|y\|_2 = 1\}.$$

The Rayleigh quotient of an arbitrary matrix $A \in C^{m\times n}$ for the singular value problem is defined by

$$\rho(x,y) = x^*Ay, \quad x \in \mathcal{X}, y \in \mathcal{Y}. \tag{1.2}$$