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ON SEMILOCAL CONVERGENCE OF INEXACT NEWTON METHODS *1)

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Abstract

Inexact Newton methods are constructed by combining Newton's method with another iterative method that is used to solve the Newton equations inexactly. In this paper, we establish two semilocal convergence theorems for the inexact Newton methods. When these two theorems are specified to Newton's method, we obtain a different Newton-Kantorovich theorem about Newton's method. When the iterative method for solving the Newton equations is specified to be the splitting method, we get two estimates about the iteration steps for the special inexact Newton methods.

Mathematics subject classification: 65H10.
Key words: Banach space, Systems of nonlinear equations, Newton's method, The splitting method, Inexact Newton methods.

1. Introduction

Consider the system of nonlinear equations

$$F(x) = 0, \tag{1.1}$$

where F: $\mathcal{D} \subset \mathcal{X} \to \mathcal{Y}$ is a continuously differentiable operator mapping an open convex subset $\mathcal{D}$ of a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$. Newton's method,

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, \cdots, \tag{1.2}$$

is the most classical method among a great deal of iterative methods used to solve (1.1). There are many research works on the existence and uniqueness of the solution of (1.1) (see [23, 24, 27]), and the convergence of Newton's method (see [12, 15, 16, 17, 20, 23, 24, 26, 27, 28] and the references therein) in addition to the classic Newton-Kantorovich theorem (see [19, 22]). These results are distinguished into two classes. One is about local convergence discussing the properties of Newton's method as x is sufficiently close to the solution x^* of (1.1) (see [24, 26, 27]), and another is about semilocal convergence which only deals with the initial point x_0 (see [15, 16, 17, 20, 23, 28]).

In order to get the $(n+1)$ iteration x_{n+1} in Newton's method, we need to solve the Newton equation

$$F'(x_n)s_n = -F(x_n), \tag{1.3}$$

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and then obtain $x_{n+1} = x_n + s_n$, see (1.2). In fact, (1.3) is a system of linear equations in the form

$$Ax = b. \tag{1.4}$$

In principle, there are two groups of methods for the solution of linear systems (1.4).

One group of methods are the so-called direct methods, or elimination methods, that is, the exact solution is determined through a finite number of arithmetic operations (in real arithmetic without considering the roundoff errors) (see [21]). It is not efficient to obtain the exact solution of (1.4) by using a direct method such as Gaussian elimination, if the coefficient matrix A is large and sparse; and when the iterate x_n is far from x^* the iteration sequence $\{x_n\}$ may not converge to x^*. We should mention that for the large sparse system of nonlinear equations, Bai and Wang [10, 11] established the following sparse factorization update algorithm based on matrix triangular factorization:

$$\begin{cases} given \quad x_0, \\ x_{n+1} = x_n + s_n, \\ U_n s_n = -H_n F(x_n), \\ H_0^{-1} U_0 \quad is \quad an \quad approximation \quad to \quad F'(x_0), \end{cases}$$

where U_n is an upper triangular matrix and H_n a unit low triangular matrix generated by two recursion formulas. As now the matrix A in (1.4) is automatically factorized into the form $A = H_n^{-1} U_n$, we can directly solve the Newton equation (1.3) by solving an upper triangular linear system of the coefficient matrix U_k at each iteration step of Newton's method.

Another group is the iterative methods, which results in the two-stage method, or sometimes termed as inner/outer iterations, for solving the system of nonlinear equations (1.1). In the two-stage method, Newton's method is the outer iteration, while an iterative method which is used to solve the Newton equations is the inner iteration. Particularly, in the two cases described below, two classes of two-stage iterative methods have been established and analyzed.

One case is that the nonlinear mapping $F(x)$ is a mildly nonlinear mapping, i.e.,

$$F(x) = Ax - \phi(x),$$

where $A \in \mathcal{R}^{n\times n}$ is nonsingular and $\phi : \mathcal{R}^n \to \mathcal{R}^n$ is a nonlinear diagonal function with certain local smoothness properties [5, 6, 8, 22]. Another case is that $F(x)$ is a linear mapping, i.e.,

$$F(x) = Ax - b,$$

where $A \in \mathcal{R}^{n\times n}$ is a large sparse and, possibly, a very ill-conditioned symmetric positive definite matrix, $x \in \mathcal{R}^n$ and $b \in \mathcal{R}^n$ (see [13]). See also [7] for efficient splitting method and its inexact variant for non-Hermitian positive definite linear systems.

When we solve the Newton equations with an iterative method, some residuals will be given. This is the reason we call this process the inexact Newton methods. To sum up, inexact Newton methods have the form

$$x_{n+1} = x_n + s_n, \quad n = 0, 1, \cdots, \tag{1.5}$$

where the step s_n satisfies

$$F'(x_n) s_n = -F(x_n) + r_n, \quad n = 0, 1, \cdots, \tag{1.6}$$

for some residual sequence $\{r_n\} \subseteq \mathcal{Y}$ (see [14]).