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A NOTE ON THE GRADIENT PROJECTION METHOD WITH EXACT STEPSIZE RULE *1)

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Abstract

In this paper, we give some convergence results on the gradient projection method with exact stepsize rule for solving the minimization problem with convex constraints. Especially, we show that if the objective function is convex and its gradient is Lipschitz continuous, then the whole sequence of iterations produced by this method with bounded exact stepsizes converges to a solution of the concerned problem.

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1. Introduction

We consider the convexly constrained minimization problem

$$\min\{f(x) \mid \; x \in \Omega\}, \tag{1}$$

where $\Omega \subseteq R^n$ is a nonempty closed convex set, and the function $f(x)$ is continuously differentiable on Ω. We say that $x^* \in \Omega$ is a stationary point of problem (1) if it satisfies condition

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \Omega, \tag{2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of R^n. Let Ω^* denote the set of all stationary points of problem (1). Then if f is convex or pseudo-convex, Ω^* becomes the solution set of this problem.

The gradient projection method was first proposed by Goldstein [5] and Levitin and Polyak [9] for solving convexly constrained minimization problems. This method is regarded as an extension of the steepest descent or Cauchy algorithm for solving unconstrained optimization problems. It now has many variants in different setting, and supplies a prototype for various more advanced projection methods. Its iterative scheme is to update $x^k \in \Omega$ according to the formula

$$x^{k+1} = P[x^k - \alpha_k \nabla f(x^k)], \tag{3}$$

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where $P[\cdot]$ denotes the projection from R^n onto Ω, i.e.,

$$P[y] = arg\min\{\|x-y\| \mid x \in \Omega\}, \qquad y \in R^n,$$

∇f is the gradient of f, and $\alpha_k > 0$ is a judiciously chosen stepsize. If α_k is taken as a global minimizer (this assumes, of course, that such a minimum exists) of the subproblem

$$\min\{f(x^k(\alpha)) \mid \alpha \geq 0\}, \tag{4}$$

where $x(\alpha) := P[x - \alpha\nabla f(x)]$, then it is called the exact stepsize. If α_k is taken as the largest element in the set $\{\gamma l^0, \gamma l^1, \gamma l^2, \cdots\}$ $(\gamma > 0, l \in (0,1))$ satisfying

$$f(x^k(\alpha_k)) \leq f(x^k) + \mu\langle\nabla f(x^k),\ x^k(\alpha_k) - x^k\rangle, \quad \mu \in (0,1), \tag{5}$$

then it is called the inexact Armijo stepsize.

The exact stepsize rule was first used by McCormick and Tapia [10] and further studied by Phelps [11, 12], Hager and Park [7], et al. Frequently, this rule is not used since it requires to evaluate the projections for all choices of $\alpha \geq 0$ while the inexact Armijo stepsize requires to only evaluate the finite projections at each iteration. However, as Hager and Park pointed out for some difficult optimization problems, where the structure of constraints is relatively simple, the exact stepsize rule is useful since it provides a mechanism for making a large step to escape from one valley of the cost function and move to another (possibly distant) valley with a smaller minimum cost. In [7], they gave an example in which the NP-hard graph partitioning problem can be formulated as a continuous quadratic programming problem whose constraints have a simple structure, and a practical procedure to evaluate the exact stepsize was given for the reformulated problem. This shows that it is quite necessary to study convergence of the gradient projection method with *exact stepsize rule*.

This paper is mainly concerned with the above issue. In Section 3 we characterize some sufficient conditions under which this method with bounded exact searches possesses some encouraging convergent properties. Especially, we obtain the result that if f is convex and ∇f is Lipschitz continuous on Ω, then the full sequence produced by the gradient projection method with bounded exact stepsizes is convergent to a solution of problem (1).

2. Some Lemmas

The analysis of the gradient projection method defined by (3) and (4) requires the following lemmas which are consequences of some basic properties of the projection operator.

Lemma 2.1. *Let P be the projection into Ω. Then for $x \in \Omega$,*
(i) $\langle x(\alpha) - x + \alpha\nabla f(x), y - x(\alpha)\rangle \geq 0$, *for all $y \in \Omega$ and $\alpha > 0$;*
(ii) $\langle\nabla f(x), x - x(\alpha)\rangle \geq \frac{1}{\alpha}\|x(\alpha) - x\|^2$, *for all $\alpha > 0$.*

Lemma 2.2. *Under the conditions of the above lemma, it holds that*
(i) $\|x(\alpha) - x\|$ *is nondecreasing on $\alpha > 0$ (see* [14]*);*
(ii) $\frac{1}{\alpha}\|x(\alpha) - x\|$ *is nonincreasing on $\alpha > 0$ (see* [4] or [3]*);*
(iii) $\langle\nabla f(x), x - x(\alpha)\rangle$ *is nondecreasing on $\alpha > 0$;*
(iv) $\frac{1}{\alpha}\langle\nabla f(x), x - x(\alpha)\rangle$ *is nonincreasing on $\alpha > 0$ (see* [15]*).*

For a given closed convex set Ω, we denote the tangent cone $T(x)$ at $x \in \Omega$ to be the closure of the cone of all feasible directions at x. Since $T(x)$ is a nonempty closed convex set in R^n,