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THE NONCONFORMING FINITE ELEMENT METHOD FOR SIGNORINI PROBLEM *1)

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Abstract

We present the Crouzeix-Raviart linear nonconforming finite element approximation of the variational inequality resulting from Signorini problem. We show if the displacement field is of H^2 regularity, then the convergence rate can be improved from $\mathcal{O}(h^{3/4})$ to quasi-optimal $\mathcal{O}(h|\log h|^{1/4})$ with respect to the energy norm as that of the continuous linear finite element approximation. If stronger but reasonable regularity is available, the convergence rate can be improved to the optimal $\mathcal{O}(h)$ as expected by the linear approximation.

Mathematics subject classification: 65N30.
Key words: Nonconforming finite element method, Signorini problem, Convergence rate.

1. Introduction

Signorini problem is one of the model problems considered in the theory of variational inequality [9, 13]. The continuous linear finite element approximations of this problem have been studied in many works, see, e.g., [2, 3, 10, 14]. As far as we have known that Scarpini and Vivaldi[14] first gave the $\mathcal{O}(h^{3/4})$ convergence rate under the condition that the displacement field u is of H^2 regularity. Then, Brezzi, Hager and Raviart[2] presented $\mathcal{O}(h)$ convergence rate by detailed analysis under the additional assumptions that $u|_{\partial\Omega} \in W^{1,\infty}(\partial\Omega)$ and that the number of points in the free boundary set where the constraint changes from binding to non-binding is finite. For simplicity, we call these points "the critical points". Later, Ben Belgacem [3] proved that under a weaker assumption, i.e., $u \in H^2(\Omega)$ and the number of critical points is finite, $\mathcal{O}(h|\log h|^{1/2})$ convergence order can be obtained. Recently, Ben Belgacem and Renard [5] established an improved result of $\mathcal{O}(h|\log h|^{1/4})$ convergence rate under the same assumptions as in [3]. However, the convergence rate is not optimal if stronger regularity and finite number of the critical points are not assumed. In this paper, we apply the Crouzeix-Raviart linear finite element[8] to approximate Signorini problem and achieve same results as those of the continuous linear finite element approximation. The whole process of the theoretical analysis is found more complicated and requires more technical treatments.

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The paper is organized as follows. In the next section, we describe the continuous setting of Signorini problem and its Crouzeix-Raviart linear finite element discretization. In section 3, we give some notations and lemmas for latter use. The main results and the corresponding proofs are provided in section 4. Finally, in the last section, numerical experiments are carried out to verify our theoretical results. Throughout this paper all the notations about Sobolev spaces can be found in [1]. We often use norm $\|\cdot\|_{0,p,\Omega}$ to represent $\|\cdot\|_{L^p(\Omega)}$ and use $\|\cdot\|_{\alpha,\Omega}$ to represent $\|\cdot\|_{H^\alpha(\Omega)}$. The semi-norm is used similarly. In addition, the frequently used constant C is a generic positive constant whose value may be different under different contexts.

2. Signorini Problem and its Finite Element Discretization

First, we state the continuous framework of Signorini problem. For the sake of simplicity, we only consider Signorini problem for the Poisson equation. The general continuous setting of this problem in $\mathbb{R}^2$ can be illustrated (a mathematical model) as follows.

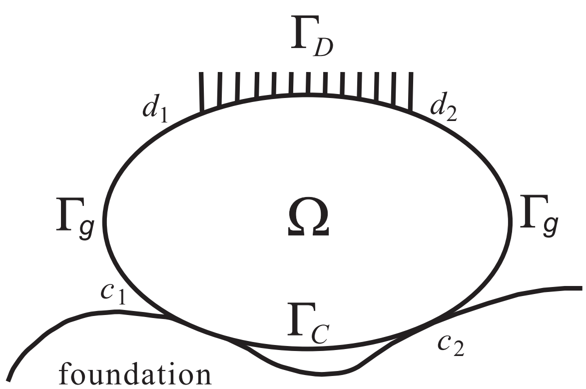

Fig. 2.1. Signorini problem.

Suppose $\Omega \subset \mathbb{R}^2$ is a Lipschitz bounded domain, which consists of three non-overlapping parts Γ_D, Γ_C and Γ_g. Here Γ_D is the fixed boundary (Dirichlet condition) with the end points d_1 and d_2, Γ_C is the contact region subjected to a rigid foundation with c_1 and c_2 as its endpoints, and Γ_g is the "glacis" with Neumann condition.

Now Signorini problem can be restated as the following mathematical model: Find

$$u \in K = \{u \in H^1_{\Gamma_D}(\Omega) : \; u \geq 0 \quad \text{a.e. on } \Gamma_C\}, \; \text{ such that}$$

$$a(u, v-u) \geq \chi(v-u), \qquad \forall\, v \in K, \tag{2.1}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad \chi(v) = \int_\Omega f\, v\, dx + \int_{\Gamma_g} g\, v\, ds.$$

The notation $H^1_{\Gamma_D} =: V$ stands for the set $\{v \in H^1(\Omega) : \quad v = 0 \text{ a.e. on } \quad \Gamma_D\}$. Moreover, $\partial\Omega = \Gamma_D \cup \Gamma_C \cup \Gamma_g$, and $int\,(\Gamma_D) \cap int\,(\Gamma_g) = \emptyset$, $\; int\,(\Gamma_C) \cap int\,(\Gamma_g) = \emptyset$ (see Fig 2.1). Here we only consider $u \geq 0$ a.e. on Γ_C instead of $u \geq \alpha$ a.e. on Γ_C in the closed convex set K,