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# A SIMPLICIAL ALGORITHM FOR COMPUTING AN INTEGER ZERO POINT OF A MAPPING WITH THE DIRECTION PRESERVING PROPERTY *1)

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**Abstract**

A mapping $f : Z^n \to R^n$ is said to possess the direction preserving property if $f_i(x) > 0$ implies $f_i(y) \geq 0$ for any integer points $x$ and $y$ with $\|x - y\|_\infty \leq 1$. In this paper, a simplicial algorithm is developed for computing an integer zero point of a mapping with the direction preserving property. We assume that there is an integer point $x^0$ with $c \leq x^0 \leq d$ satisfying that $\max_{1 \leq i \leq n}(x_i - x_i^0)f_i(x) > 0$ for any integer point $x$ with $f(x) \neq 0$ on the boundary of $H = \{x \in R^n \mid c - e \leq x \leq d + e\}$, where $c$ and $d$ are two finite integer points with $c \leq d$ and $e = (1, 1, \cdots, 1)^\top \in R^n$. This assumption is implied by one of two conditions for the existence of an integer zero point of a mapping with the preserving property in van der Laan et al. (2004). Under this assumption, starting at $x^0$, the algorithm follows a finite simplicial path and terminates at an integer zero point of the mapping. This result has applications in general economic equilibrium models with indivisible commodities.

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*Key words:* Integer Zero Point, Direction Preserving, Simplicial Algorithm, Triangulation, Existence.

## 1. Introduction

The problem we consider in this paper is to compute an integer zero point of a mapping $f : Z^n \to R^n$. The interests in integer zero points or fixed points of a mapping have been inspired by the work in Iimura (2003) though the statement of the existence of a discrete fixed point in Iimura (2003) is incorrect and a corrected statement was given in Iimura et al. (2004) after an application of the integrally convex set defined in Favati and Tardella (1990). A brief introduction to the applications of discrete fixed points of a mapping in economics can be found in Iimura (2003) and references therein.

Following the definition in Iimura (2003), we say that $f(x)$ satisfies the direction preserving property if $f_i(x) > 0$ implies $f_i(y) \geq 0$ for any integer points $x$ and $y$ with $\|x - y\|_\infty \leq 1$. We assume throughout this paper that $f(x)$ satisfies the direction preserving property. Recently, under two different conditions, based on the $2n$-ray algorithm in van der Laan and Talman (1981), a constructive proof of the existence of an integer zero point of a mapping with the direction preserving property was obtained in van der Laan et al. (2004). Those two conditions can be stated as follows.

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**Condition 1.1.** *There exist integer vectors $m$, $x^0$, and $M$ with $m + e < x^0 < M - e$ and $e = (1, 1, \cdots, 1)^\top$ such that $(x - x^0)^\top f(x) > 0$ for any integer point $x$ on the boundary of $C = \{y \in R^n \mid m \leq y \leq M\}$.*

**Condition 1.2.** *There exists an integer vector $u$ with $u > e$ such that $f_k(x)f_k(-y) \leq 0$, $k = 1, 2, \cdots, n$, for any two cell connected integer points $x$ and $y$ on the boundary of $U = \{z \in R^n \mid -u \leq z \leq u\}$.*

Given these two conditions, the following two theorems can be found in van der Laan et al. (2004).

**Theorem 1.1.** *If Condition 1.1 holds, there exists an integer point $x^* \in C$ such that $f(x^*) = 0$.*

**Theorem 1.2.** *If Condition 1.2 holds, there exists an integer point $x^* \in U$ such that $f(x^*) = 0$.*

In Dang (2005), a new condition for the existence of an integer zero point of the mapping was introduced, which is as follows.

**Condition 1.3.** *There is an integer point $x^0$ with $c \leq x^0 \leq d$ satisfying that $\max_{1 \leq i \leq n}(x_i - x_i^0)f_i(x) > 0$ for any integer point $x$ with $f(x) \neq 0$ on the boundary of $H = \{x \in R^n \mid c - e \leq x \leq d + e\}$, where $c$ and $d$ are two finite integer points with $c \leq d$ and $e = (1, 1, \cdots, 1)^\top$.*

For Conditions 1.1, 1.2 and 1.3, the following lemma was proved in Dang (2005).

**Lemma 1.1.** *Condition 1.1 implies Condition 1.3. However, Condition 1.3 implies neither Condition 1.1 nor Condition 1.2.*

Given Condition 3, based on the $(n+1)$-ray algorithm proposed in van der Laan and Talman (1979), the following theorem was proved in Dang (2005).

**Theorem 1.3.** *If Condition 1.3 holds, there exists an integer point $x^* \in H$ such that $f(x^*) = 0$.*

In this paper, based on the 2-ray algorithm given in Yamamoto (1984), we will develop a simplicial algorithm for computing an integer zero point of the mapping satisfying Condition 3. The 2-ray algorithm is one of simplicial methods for computing a fixed point of a continuous mapping. The simplical methods were originated in Scarf (1967), and have been substantially developed after Scarf's seminal work (e.g., Allgower and Georg, 2000; Dang, 1991, 1995; Dang and Maaren, 1998; Eaves, 1972; Eaves and Saigal, 1972; Forster, 1995; Kojima and Yamamoto, 1982; Kuhn, 1968; van der laan and Talman, 1979, 1981; Merrill, 1972; Scarf, 1973, 1981; Todd, 1976; Yamamoto, 1983; etc.). The basic idea of the algorithm is as follows. It assigns to each integer point of $H$ an integer label and subdivides $H$ into integer simplices. Starting at $x^0$, the algorithm follows a finite simplicial path and terminates at an integer zero point of the mapping. An advantage of the 2-ray algorithm over the $(n+1)$-ray algorithm is that some more superior triangulations of $R^n$ can be its underlying triangulations without any modifications.

The rest of this paper is organized as follows. An integer labeling rule is introduced in Section 2. The algorithm is given in Section 3.