Journal of Computational Mathematics, Vol. 24, No. 6, 2006, 693–710.

ALTERNATING PROJECTION BASED PREDICTION-CORRECTION METHODS FOR STRUCTURED VARIATIONAL INEQUALITIES *1)

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Abstract

The monotone variational inequalities $\mathrm{VI}(\Omega, F)$ have vast applications, including optimal controls and convex programming. In this paper we focus on the VI problems that have a particular splitting structure and in which the mapping F does not have an explicit form, therefore only its function values can be employed in the numerical methods for solving such problems. We study a set of numerical methods that are easily implementable. Each iteration of the proposed methods consists of two procedures. The first (prediction) procedure utilizes alternating projections to produce a predictor. The second (correction) procedure generates the new iterate via some minor computations. Convergence of the proposed methods is proved under mild conditions. Preliminary numerical experiments for some traffic equilibrium problems illustrate the effectiveness of the proposed methods.

Mathematics subject classification: 65K10, 90C25, 90C30.
Key words: Structured variational inequality, Monotonicity, Prediction-correction method.

1. Introduction

A variational inequality problem, denoted by $\mathrm{VI}(\Omega, F)$, is to find a vector $u^* \in \Omega$ such that

$$(u - u^*)^T F(u^*) \geq 0, \quad \forall\, u \in \Omega, \tag{1.1}$$

where Ω is a nonempty closed convex subset of $\mathbb{R}^l$, and F is a mapping from $\mathbb{R}^l$ into itself. In this paper, we consider the VI problem with the following structure:

$$(x^*, y^*) \in \mathcal{D}, \quad \begin{cases} (x - x^*)^T f(x^*) \geq 0, \\ (y - y^*)^T g(y^*) \geq 0, \end{cases} \qquad \forall\, (x, y) \in \mathcal{D}, \tag{1.2}$$

where

$$\mathcal{D} = \{(x, y) | x \in \mathcal{X},\ y \in \mathcal{Y},\ Ax + By = b\}, \tag{1.3}$$

$\mathcal{X}$ and $\mathcal{Y}$ are given nonempty closed convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively, $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times p}$ are given matrices, $b \in \mathbb{R}^m$ is a given vector, $f : \mathcal{X} \to \mathbb{R}^n$ and $g : \mathcal{Y} \to \mathbb{R}^p$ are monotone operators. Problem (1.2)-(1.3) is a special case of the general VI problem (1.1), which

* Received December 9, 2005.
1) The research of the first author is supported by NSFC Grant 10571083. The research of the second author is supported in part by grants from Hong Kong Baptist University and the Research Grant Council of Hong Kong.

has numerous important applications, including applications in the fields of optimal controls and convex programming (see [1, 6, 7]).

Since in practice such problems usually involve large number of variables, numerical methods that can make use of the decomposed structure of problem (1.2)-(1.3) can greatly save computer storage as well as computing time. A number of decomposition methods have been proposed, for examples, see [3, 4, 5, 7, 8, 9, 15].

In many applications, the mapping f (resp. g) cannot be expressed explicitly and for a given $x \in \mathcal{X}$ (resp. $y \in \mathcal{Y}$), the function value $f(x)$ (resp. $g(y)$) can only be obtained via certain procedures. Given a variable value, the evaluation of f or g can be costly and time-consuming, and sometimes may pose social or political impact (such as posing toll charges to evaluate the traffic flow), therefore should not be taken lightly. In such applications, efficient numerical methods which only employ function values are highly desired.

Among all the existing decomposition methods which achieve linear convergence, in each iteration a subproblem equivalent to an implicit projection calls to be solved, as illustrated below. Solving each subproblem usually requires numerous function evaluations. In this paper we present a set of decomposition methods that involve only explicit projections, therefore require only one function evaluation in each iteration, yet they also yield linear convergence. The numerical experiments presented in Section 6 illustrate the effectiveness of the methods.

The proposed methods are motivated by the existing proximal alternating directions methods (abbreviated as PADMs) proposed in [15]. We briefly describe the PADMs as follows: First, by attaching a Lagrange multiplier vector $\lambda \in \mathbb{R}^m$ to the linear constraint $Ax + By = b$, the VI problem (1.2)-(1.3) is converted into the following equivalent non-constrained form:

$$(x^*, y^*, \lambda^*) \in \mathcal{W}, \quad \begin{cases} (x - x^*)^T (f(x^*) - A^T\lambda^*) \geq 0, \\ (y - y^*)^T (g(y^*) - B^T\lambda^*) \geq 0, \\ (\lambda - \lambda^*)^T (Ax^* + By^* - b) \geq 0, \end{cases} \quad \forall\, (x, y, \lambda) \in \mathcal{W} \tag{1.4}$$

where

$$\mathcal{W} = \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m. \tag{1.5}$$

We denote VI problem (1.4)-(1.5) by VI($\mathcal{W}, Q$), where

$$Q(w) = Q(x, y, \lambda) = \begin{pmatrix} f(x) - A^T\lambda \\ g(y) - B^T\lambda \\ Ax + By - b \end{pmatrix}. \tag{1.6}$$

Given a triplet $w^k = (x^k, y^k, \lambda^k) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m$, the PADMs generate a new iterate $\tilde{w}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{\lambda}^k) \in \mathcal{X} \times \mathcal{Y} \times \mathbb{R}^m$ via the following general procedure:
Given $(x^k, y^k, \lambda^k) \in \mathcal{W}$, first find an $\tilde{x}^k \in \mathcal{X}$ such that

$$(x' - \tilde{x}^k)^T \{f(\tilde{x}^k) - A^T[\lambda^k - \beta(A\tilde{x}^k + By^k - b)] + r(\tilde{x}^k - x^k)\} \geq 0, \quad \forall\, x' \in \mathcal{X}. \tag{1.7}$$

Then find a $\tilde{y}^k \in \mathcal{Y}$ such that

$$(y' - \tilde{y}^k)^T \{g(\tilde{y}^k) - B^T[\lambda^k - \beta(A\tilde{x}^k + B\tilde{y}^k - b)] + s(\tilde{y}^k - y^k)\} \geq 0, \quad \forall\, y' \in \mathcal{Y}. \tag{1.8}$$

Finally, update $\tilde{\lambda}^k$ via

$$\tilde{\lambda}^k = \lambda^k - \beta(A\tilde{x}^k + B\tilde{y}^k - b). \tag{1.9}$$

Here $\beta > 0$ is a given *penalty parameter* of the linear constraint $Ax + By - b = 0$. The coefficients $r > 0$ and $s > 0$ in formulas (1.7) and (1.8) respectively are referred to as *proximal parameters*. The method is convergent by taking $w^{k+1} = \tilde{w}^k$ (for a proof see [12]).