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MODIFIED NEWTON'S ALGORITHM FOR COMPUTING THE GROUP INVERSES OF SINGULAR TOEPLITZ MATRICES *1)

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Abstract

Newton's iteration is modified for the computation of the group inverses of singular Toeplitz matrices. At each iteration, the iteration matrix is approximated by a matrix with a low displacement rank. Because of the displacement structure of the iteration matrix, the matrix-vector multiplication involved in Newton's iteration can be done efficiently. We show that the convergence of the modified Newton iteration is still very fast. Numerical results are presented to demonstrate the fast convergence of the proposed method.

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Key words: Newton's iteration, Group inverse, Toeplitz matrix, Displacement rank.

1. Introduction

Let A be an $n \times n$ Toeplitz matrix [9, 12, 13], i.e.,

$$A = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{2-n} & a_{1-n} \\ a_1 & a_0 & a_{-1} & \ddots & a_{2-n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix}.$$

The main aim of this paper is to modify Newton's iteration for the computation of the group inverse of A if A is singular with index 1, i.e.,

$$\mathrm{rank}(A) = \mathrm{rank}(A^2) < n.$$

The group inverse of A is the unique solution of the following three equations [1, 4, 14]

$$A^2X = A \quad XAX = X \quad \text{and} \quad AX = XA$$

and we denote it A_g throughout the paper. The application of the group inverses of matrices can be found in the field of Markov chains [4] and numerical analysis [5, 15, 16, 18, 19]. The

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computation of the inverses and the Moore-Penrose inverses of structured matrices [3, 2, 17] is a recent interesting problem in matrix computation, see for instance [6].

In this paper, Newton's iteration is modified for the computation of the group inverses of singular Toeplitz matrices. At each iteration, the iteration matrix is approximated by a matrix with a low displacement rank. Because of the displacement structure of the iteration matrix, the matrix-vector multiplication involved in Newton's iteration can be done efficiently. We show that the convergence of the modified Newton iteration is still very fast.

The outline of this paper is as follows. In Section 2, we review the concept of ϵ-displacement rank [3]. In Section 3, we introduce the modified Newton iteration and show the convergence results. Numerical results in Section 4 are presented to demonstrate the fast convergence of the proposed method. In Section 5, we present some concluding remarks.

2. ϵ-Displacement Rank

2.1 Displacement Rank

The concept of displacement rank was introduced by Kailath and his coauthors (see [10]) for close-to-Toeplitz matrices and was systematically studied in the general case in [9]. There are many definitions for displacement rank. Here we briefly describe two of them that will be used in the following discussion. Let us denote

$$C^+ = \begin{bmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbf{R}^{n\times n}, \qquad C^- = \begin{bmatrix} 0 & & & -1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbf{R}^{n\times n}.$$

The displacements for an n-by-n matrix B can be defined by

$$\Delta^+(B) = C^+B - BC^-, \quad \Delta^-(B) = C^-B - BC^+.$$

The rank of $\Delta^+(B)$ (or $\Delta^-(B)$) is called the $(+)$-displacement (or $(-)$-displacement) rank of B and are denoted by $drk^+(B)$ (or $drk^-(B)$). For a Toeplitz matrix A, we have $drk^+(A) \le 2$ and $drk^-(A) \le 2$, see for instance [9].

The operators $\Delta^+(\cdot)$ and $\Delta^-(\cdot)$ are both invertible. If we know the displacement of B, then B can be recovered by the sum of a series of products of circulant matrix and anti-circulant matrices. Let $\mathbf{h} = [h_1, h_2, \cdots, h_n]^T$ and $C^+(\mathbf{h})$ be a circulant matrix with its first column $\mathbf{h}$:

$$C^+(\mathbf{h}) = \begin{bmatrix} h_1 & h_n & \cdots & h_3 & h_2 \\ h_2 & h_1 & h_n & \ddots & h_3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ h_{n-1} & \ddots & h_2 & h_1 & h_n \\ h_n & h_{n-1} & \cdots & h_2 & h_1 \end{bmatrix}$$

and $C^-(\mathbf{h})$ be an anti-circulant matrix with its first column $\mathbf{h}$:

$$C^-(\mathbf{h}) = \begin{bmatrix} h_1 & -h_n & \cdots & -h_3 & -h_2 \\ h_2 & h_1 & -h_n & \ddots & -h_3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ h_{n-1} & \ddots & h_2 & h_1 & -h_n \\ h_n & h_{n-1} & \cdots & h_2 & h_1 \end{bmatrix}.$$