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CHEBYSHEV WEIGHTED NORM LEAST-SQUARES SPECTRAL METHODS FOR THE ELLIPTIC PROBLEM *1)

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Abstract

We develop and analyze a first-order system least-squares spectral method for the second-order elliptic boundary value problem with variable coefficients. We first analyze the Chebyshev weighted norm least-squares functional defined by the sum of the L^2_w- and H^{-1}_w- norm of the residual equations and then we replace the negative norm by the discrete negative norm and analyze the discrete Chebyshev weighted least-squares method. The spectral convergence is derived for the proposed method. We also present various numerical experiments. The Legendre weighted least-squares method can be easily developed by following this paper.

Mathematics subject classification: 65F10, 65F30.
Key words: Least-squares methods, Spectral method, Negative norm.

1. Introduction

Let Ω be the square $(-1,1)^2$. We consider the second-order elliptic boundary value problem:

$$\begin{cases} -\nabla\cdot A\nabla p + \mathbf{b}\cdot\nabla p + c_0\, p &= f, \quad \text{in } \Omega,\\ p &= 0, \quad \text{on } \Gamma_D,\\ \mathbf{n}\cdot A\nabla p &= 0, \quad \text{on } \Gamma_N, \end{cases} \tag{1.1}$$

where $\partial\Omega = \Gamma_D \cup \Gamma_N$ denotes the boundary of Ω, A is a 2×2 symmetric matrix of bounded functions, f is a given continuous function, $\mathbf{b}$ is a bounded vector function and c_0 is a given bounded function, and $\mathbf{n}$ is the outward unit vector normal to the boundary. We assume that the matrix A is uniformly elliptic such as

$$0 < \lambda\, \boldsymbol{\xi}^t\boldsymbol{\xi} \le \boldsymbol{\xi}^t A(x,y)\boldsymbol{\xi} \le \Lambda\, \boldsymbol{\xi}^t\boldsymbol{\xi} < \infty \tag{1.2}$$

for all $\boldsymbol{\xi}\in\Re^2$ and almost all $(x,y)\in\bar{\Omega}$.

In recent years there has been lots of interest in the use of first-order system least-squares method (FOSLS) for numerical approximations of elliptic partial differential equations and systems. Introducing an extra physical meaningful variable, for example the flux $\mathbf{u} = A\nabla p$, the equation (3.1) can be written as an equivalent first-order system of partial differential equations and then one may try to apply mixed methods to find the approximation solution. Among the several mixed methods the least-squares methods have several benefits such that the resulting algebraic system is always positive symmetric and the methods can avoid LBB compatibility condition. For more details we refer to [2] and references therein.

In [4] and [5], Cai, Lazarov, Manteuffel and McCormick developed an L^2-norm least squares for scalar second order elliptic partial differential equations. But the limitation of such L^2-norm FOSLS is the requirement of sufficient smoothness of the underlying problem which guarantees

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the equivalence of norms between $H(\mathrm{div};\Omega) \cap H(\mathrm{curl}A;\Omega)$ and $H^1(\Omega)^d$, where $d = 2$ or 3, so that it can be approximated by standard continuous finite element space. But those two spaces are not equivalent in general when the domain Ω is not smooth or not convex, or the coefficient is not continuous. To overcome such a limitation, in [7] Cai and Shin developed the discrete FOSLS by directly approximating $H(\mathrm{div}) \cap H(\mathrm{curl})$-type space based on the Helmholtz decomposition, in which under general assumptions they established error estimates in the L^2- and H^1-norms for the vector and scalar variables, respectively. Also, in [3] Bramble, Lazarov and Pasciak developed the discrete negative norm FOSLS for overcoming the same limitation. An alternative using adjoint FOSLS, so-called FOSLL*, can be found in [6].

In this paper, instead of using finite element method, we adopt Legendre and Chebyshev spectral method to solve the first-order system corresponding to (3.1) in the least-squares approaches. The usage of spectral methods instead of using piecewise polynomials in a finite element approach has been known as very accurate and popular regardless of higher amount of work for approximating solutions of various kinds of differential equations (see [1], [13]). In [9], Kim, Lee and Shin recently developed the fusion method combining the concept of L^2 FOSLS and spectral collocation method, so-called least-squares spectral collocation method, to solve second order elliptic boundary value problems with constant coefficients. They derived the spectral error estimates which allow us to get the spectral convergence for both Legendre and Chebyshev approximations.

In order to apply least-squares spectral method to general elliptic problems having various coefficients, we first develop the Chebyshev weighted negative norm least-squares method and then we develop the discrete Chebyshev weighted negative norm least-squares method. The negative norm least-squares given in [3] is further extended to the Chebyshev weighted negative norm least-squares method. To define the discrete weighted negative norm least-squares functional under the polynomial space, we first define a discrete solution operator $T_N : H_w^{-1}(\Omega) \to \mathcal{Q}_N^0$, where $\mathcal{Q}_N$ is the space consisting of all polynomials of degree less than or equal to N and $\mathcal{Q}_N^0 = \mathcal{Q}_N \cap H_{0,w}^1(\Omega)$, and then we establish the relation between $\|f\|^2_{H_w^{-1}(\Omega)}$ and $(f, T_N f)_w$. Owing to such estimates we show the equivalence between the proposed homogeneous discrete least-squares functional $G_{w,N}(\mathbf{v}, q; 0)$ and the product norm $|||(\mathbf{v}, q)|||_w^2 := \|\mathbf{v}\|^2_{L^2_w(\Omega)} + N^{-1}\|\nabla \cdot \mathbf{v}\|^2_{L^2_w(\Omega)} + \|p\|^2_{H^1_w(\Omega)}$ over $H_w(\mathrm{div};\Omega) \times H^1_{0,w}(\Omega)$. The equivalence is more improved than that of the result given in [3], in which the norm is given by $\|\mathbf{v}\|_{L^2_w(\Omega)} + \|q\|_{H^1_w(\Omega)}$. We also establish the spectral convergence such that for $(\mathbf{u}, p) \in H_w^{s-1}(\Omega)^2 \times H_w^s(\Omega)$ $(s \geq 2)$

$$|||(\mathbf{u} - \mathbf{u}_N, p - p_N)|||_w \leq C\, N^{1-s}(\|\mathbf{u}\|_{H_w^{s-1}(\Omega),w} + \|p\|_{H_w^s(\Omega)}).$$

The spectral Galerkin approximation using least-squares principle developed in this paper is slightly different from the spectral collocation method in the respect of using the $L^2_w(\Omega)$ scalar product instead of the discrete scalar product. The exact computation of spectral Galerkin approach is somewhat complicated. But, if the coefficients are continuous then it can be further approximated by using Gaussian quadrature formula. That is, the $L^2_w(\Omega)$ scalar product of continuous functions can be approximated by the discrete scalar product using Chebyshev-Gauss-Lobatto(CGL) points and the corresponding quadrature weights. Also, based on this paper one may develop the spectral element approximation or high-order element method like *hp*-method for the general elliptic problems and more complicated problems, e.g., Stokes problems and Navier-Stokes problems.

Throughout this paper, we assume that w is the Chebyshev weight function, i.e., we will investigate our theory for only Chebyshev spectral approximation. For Legendre case, one may easily obtain the similar results following the arguments of this paper.

The paper is organized as follows. The definitions, notations and basic facts are presented in the following section 2. The weighted negative norm first-order system least-squares method is introduced and analyzed by showing ellipticity and continuity in section 3. And the spectral Galerkin approximation for the discrete least-squares is discussed with convergence analysis in