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SUPERCONVERGENCE AND A POSTERIORI ERROR ESTIMATES FOR BOUNDARY CONTROL GOVERNED BY STOKES EQUATIONS *1)

Hui-po Liu   Ning-ning Yan
(*Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China*)

Dedicated to the 70th birthday of Professor Lin Qun

Abstract

In this paper, the superconvergence results are derived for a class of boundary control problems governed by Stokes equations. We derive superconvergence results for both the control and the state approximation. Base on superconvergence results, we obtain asymptotically exact a posteriori error estimates.

Mathematics subject classification: 49J20, 65N30.
Key words: Boundary control, Finite element method, Superconvergence, A posteriori error estimates.

1. Introduction

Finite element approximation of optimal control problems plays a very important role in the numerical methods for these problems. The literature in this aspect is huge. A priori error estimates of finite element approximation were established for the distributed optimal control of problems governed by partial differential equations; see, for example, [6] and [10]. A posteriori error estimates of some distributed optimal control problems has been studied, see, [18], [19], [20] and [11]. As one of important kinds of optimal control problems, the boundary control problem is widely used in scientific and engineering computing. A priori error estimates have been provided for linear boundary control problems, see [7] and [9]. A posteriori error estimates have also been obtained for boundary control problems, see [16] and [17]. In recent years, the superconvergence property of some distributed optimal control problems including the distributed optimal control problem governed by Stokes equations have been investigated, see, for example, [5], [12], [15], [24]. Although superconvergence property of finite element approximation is widely used in numerical simulations, it is not yet been utilized in boundary control problems.

In this work, we present the superconvergence analysis and a posteriori error estimates for the finite element approximation of the boundary control problems governed by Stokes equations. The purpose of this work is to derive superconvergence results of the control and the state for the boundary control problems governed by Stokes equations. Based on superconvergence results, we obtain asymptotically exact a posteriori error estimates. The obtained error estimates can then be used as a posteriori error indicators to construct reliable adaptive finite element methods.

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The outline of this paper is as follows: In section 2, we shall give a weak formula for the boundary control problem and then discuss the finite element approximation of the control problem. In section 3, a superconvergence result for the control $\mathbf{u}$ is derived by applying patch recovery operator and the supercovergence analysis technique. In section 4, recovery and superconvergence for state and co-state are derived by using L^2 projection methods. In section 5, recovery type a posteriori error estimates are derived. In the last section, we discuss briefly some possible future work.

Let Ω be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega$. In this paper we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|.\|_{m,q,\Omega}$ and the seminorm $|.|_{m,q,\Omega}$. We shall extend these (semi)norms to vector functions whose components belong to $W^{m,q}(\Omega)$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. We denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)(H_0^m(\Omega))$ equipped with the norm $\|.\|_{m,\Omega}$ and the seminorm $|.|_{m,\Omega}$. In addition, c or C denotes a general positive constant independent of h.

2. Finite Element Approximation of Boundary Control Problems

In this section, we study the finite element approximation of the boundary control problems governed by Stokes equations, where the boundary control is applied on the part of the boundary, and the fixed boundary condition is given on the other part of the boundary. In the rest of the paper, let Ω be a bounded open set in $\mathbb{R}^2$ with boundaries $\partial\Omega$, where $\partial\Omega = \Gamma_a \cup \Gamma_b$, $\Gamma_a \cap \Gamma_b = \emptyset$, $\text{meas}(\Gamma_a) > 0$ and $\text{meas}(\Gamma_b) > 0$. We shall take the state space $\mathbf{V} = \{\mathbf{v} \in (H^1(\Omega))^2 : \mathbf{v}|_{\Gamma_a} = 0\}$, the control space $\mathbf{U} = (L^2(\Gamma_b))^2$, and the observation space $\mathbf{Y} = (L^2(\Gamma_b))^2$. Let B be a linear continuous operator from $\mathbf{U}$ to $\mathbf{U}$. Let $\mathbf{K} = \{\mathbf{u} \in \mathbf{U} : \mathbf{u} \geq 0\}$. We are interested in the following boundary control problem: Give $\mathbf{f}$, $\mathbf{y}_d$, $\mathbf{z}_b$, find $(\mathbf{y},\mathbf{u}) \in \mathbf{V}\times\mathbf{K}$ such that

$$\begin{aligned} \min_{\mathbf{u}\in\mathbf{K}}\{\frac{1}{2}\|\mathbf{y}-\mathbf{y}_d\|^2_{L^2(\Gamma_b)} + \frac{1}{2}\|\mathbf{u}\|^2_{L^2(\Gamma_b)}\}&, \\ -\triangle\mathbf{y} + \nabla r = \mathbf{f} \quad \text{in } \Omega&, \\ \text{div}\mathbf{y} = 0 \quad \text{in } \Omega&, \\ \mathbf{y} = 0 \quad \text{on } \Gamma_a&, \\ \frac{\partial\mathbf{y}}{\partial\mathbf{n}} - r\mathbf{n} = \mathbf{z}_b + B\mathbf{u} \quad \text{on } \Gamma_b&, \end{aligned} \tag{2.1}$$

where $\mathbf{f} \in (L^2(\Omega))^2$, $\mathbf{y}_d, \mathbf{z}_b \in (L^2(\Gamma_b))^2$. Let $\mathbf{H} = (L^2(\Omega))^2$, $Q = L^2(\Omega)$, and let

$$\begin{aligned} a(\mathbf{y},\mathbf{w}) &= \int_\Omega \nabla\mathbf{y}\cdot\nabla\mathbf{w}, \quad \forall \mathbf{y},\mathbf{w} \in \mathbf{V}, \\ b(\mathbf{v},r) &= \int_\Omega r\text{div}\mathbf{v} \quad \forall(\mathbf{v},r) \in \mathbf{V}\times Q, \\ (\mathbf{f},\mathbf{w}) &= \int_\Omega \mathbf{f}\mathbf{w}, \quad \forall(\mathbf{f},\mathbf{w}) \in \mathbf{H}\times\mathbf{V}, \\ (\mathbf{u},\mathbf{v})_\mathbf{U} &= \int_{\Gamma_b} \mathbf{u}\mathbf{v}, \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{U}. \end{aligned}$$

Then the weak formula of the state equation reads: find $(\mathbf{y}(\mathbf{u}), r(\mathbf{u})) \in \mathbf{V}\times Q$ such that

$$\begin{aligned} a(\mathbf{y}(\mathbf{u}),\mathbf{w}) - b(\mathbf{w}, r(\mathbf{u})) = (\mathbf{f},\mathbf{w}) + (B\mathbf{u}+\mathbf{z}_b,\mathbf{w})_\mathbf{U} \quad &\forall \mathbf{w}\in\mathbf{V}. \\ b(\mathbf{y}(\mathbf{u}),\phi) = 0 \quad &\forall\phi\in Q. \end{aligned}$$

For the above problem, it is well known that the following Babuška-Brezzi condition holds (see [8]):

$$\sup_{\mathbf{v}\in\mathbf{V}} \frac{b(\mathbf{v},q)}{\|\mathbf{v}\|_{1,\Omega}} \geq C\|q\|_{0,\Omega} \quad \forall q \in Q,$$