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ARTIFICIAL BOUNDARY METHOD FOR BURGERS' EQUATION USING NONLINEAR BOUNDARY CONDITIONS *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

This paper discusses the numerical solution of Burgers' equation on unbounded domains. Two artificial boundaries are introduced and boundary conditions are obtained on the artificial boundaries, which are in nonlinear forms. Then the original problem is reduced to an equivalent problem on a bounded domain. Finite difference method is applied to the reduced problem, and some numerical examples are given to show the effectiveness of the new approach.

Mathematics subject classification: 65M06, 65M12, 65M15.
Key words: Burgers' equation, Unbounded domain, Artificial boundary condition.

1. Introduction

For numerical solutions of partial differential equations on unbounded domains, the artificial boundary method is the most efficient method and has been applied to many application problems [10, 16, 11, 5, 19, 15, 7, 8]. The artificial boundary method generally means to introduce artificial boundaries, find boundary conditions on the artificial boundaries, and reduce the original problem to an equivalent or approximate problem defined on a bounded domain. In general, the boundary conditions on the artificial boundaries are obtained by considering the exterior problems outside the artificial boundaries. In most cases, the basic assumption of the artificial boundary method is that the equation is linear. Then analytic forms of the boundary conditions on the artificial boundaries can be obtained. Usually, the artificial boundary method can not be applied directly to nonlinear problems. However, for some problems, if the equation can be linearized outside the artificial boundaries, then it is possible to find the boundary conditions on the artificial boundaries [9, 14, 6].

In this paper, we consider the numerical solution of the Burgers' equation on unbounded domains

$$u_t + uu_x - \nu u_{xx} = f(x,t), \quad \forall (x,t) \in R \times (0,T], \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

$$u(x,t) \to 0, \quad |x| \to +\infty, \tag{1.3}$$

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where $\nu > 0$ is viscous coefficient, the source term $f(x,t)$ and initial data $u_0(x)$ have compact supports satisfying

$$\text{supp}\{f(x,t)\} \subset [x_l, x_r] \times [0,T], \quad \text{supp}\{u_0(x)\} \subset [x_l, x_r].$$

Burgers' equation is a simple, but important model in fluid dynamics. Burgers' equation itself models various kind of physical phenomena such as turbulence [3]. Besides, due to its similarity to the Navier-Stokes equation, the solution of Burgers' equation is a natural first step towards the designing of new numerical methods for flow problems. When the domain is bounded, various numerical approaches have been discussed by many researches (see recent papers [2, 4] and references therein). In this paper, we mainly deal with the difficulty of the unboundness of the solution domain. The idea is to introduce artificial boundaries to make the computational domain finite, and find boundary conditions on the artificial boundaries. Unlike linear problems, these boundary conditions are nonlinear. Then we solve the problem on the finite domain, the reduced problem is equivalent to the original problem in the sense that the solution is the same as the restriction of the original problem. In section 2, we describe the artificial boundary method using nonlinear boundary conditions. In section 3, we consider the numerical approximation of the reduced problem. In section 4, we give some numerical examples to show the effectiveness of the new approach.

2. The Artificial Boundary Method

Consider the problem (1.1)-(1.3). We introduce two artificial boundaries

$$\Gamma_l = \{(x,t)|\ x = x_l, 0 \le t \le T\},$$
$$\Gamma_r = \{(x,t)|\ x = x_r, 0 \le t \le T\}.$$

Then the unbounded domain $\Omega = R \times [0,T]$ is divided into three parts (see Fig. 1),

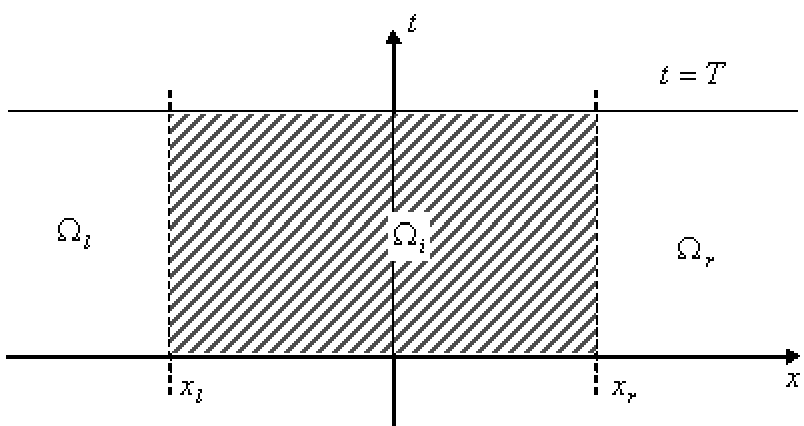

Fig. 1. The artificial boundaries

$$\Omega_l = \{(x,t)|\ x \le x_l, 0 \le t \le T\},$$
$$\Omega_r = \{(x,t)|\ x \ge x_r, 0 \le t \le T\},$$
$$\Omega_i = \{(x,t)|\ x_l < x < x_r, 0 \le t \le T\},$$

where Ω_i is the computational domain and its two boundary conditions at $x = x_l$ and $x = x_r$ are to be determined.