Journal of Computational Mathematics, Vol.24, No.3, 2006, 227–238.

LOCAL AND PARALLEL FINITE ELEMENT ALGORITHMS FOR THE NAVIER-STOKES PROBLEM *1)

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Dedicated to the 70th birthday of Professor Lin Qun

Abstract

Based on two-grid discretizations, in this paper, some new local and parallel finite element algorithms are proposed and analyzed for the stationary incompressible Navier-Stokes problem. These algorithms are motivated by the observation that for a solution to the Navier-Stokes problem, low frequency components can be approximated well by a relatively coarse grid and high frequency components can be computed on a fine grid by some local and parallel procedure. One major technical tool for the analysis is some local a priori error estimates that are also obtained in this paper for the finite element solutions on general shape-regular grids.

Mathematics subject classification: 65N15, 65N30, 76M10, 76D07.
Key words: Navier-Stokes problem, Finite element, Two-grid method, Local and parallel algorithm.

1. Introduction

In this paper, we will propose some new parallel techniques for finite element computations of the stationary incompressible Navier-Stokes problem. These techniques are based on our understanding of the local and global properties of a finite element solution to the Navier-Stokes problem. Simply speaking, the global behavior of a solution is mostly governed by low frequency components while the local behavior is mostly governed by high frequency components. The main idea of our new algorithms is to use a coarse grid to approximate the low frequencies and then to use a fine grid to correct the resulted residual (which contains mostly high frequencies) by some local and parallel procedures. One technical tool for motivating this idea is the local error estimate for finite element approximations. Let (w_h, r_h) be a finite element approximation to the linearized Navier-Stokes problem on a quasi-uniform grid $T^h(\Omega)$, then the following kind of local estimate holds (see Lemma 3.2):

$$\|w_h\|_{1,D} + \|r_h\|_{0,D} \le c(\|w_h\|_{0,\Omega_0} + \|r_h\|_{-1,\Omega_0} + \|f\|_{-1,\Omega_0}), \tag{1.1}$$

* Received March 1, 2006.
1) The first author was partially subsidized by the NSF of China 10371095. The third author was partially supported by the National Science Foundation of China under the grant 10425105 and the National Basic Research Program under the grant 2005CB321704.

where $D \subset\subset \Omega_0 \subset \Omega$, here $D \subset\subset \Omega_0$ means $\mathrm{dist}(\partial D \setminus \partial\Omega, \partial\Omega_0 \setminus \partial\Omega) > 0$.

This paper may be considered as a sequel of papers [7,23,24,25] on designing local and parallel finite element algorithms. In [7,23,24,25], local and parallel algorithms for a class of elliptic problems and the Stokes problem are studied, based on the local behaviors of finite element approximations on sharp-regular grids.

The rest of the paper is organized as follows. In the coming section, some preliminary materials and assumptions of mixed finite element spaces are provided. In section 3, a number of local a priori error estimates are obtained for the finite element discretization of the Navier-Stokes problem. Based upon these local error estimates, several new local and parallel algorithms are devised and analyzed in section 4.

2. Preliminaries

In this section, we shall describe some basic notations and assumptions on the mixed finite element spaces and then study some properties of the mixed finite element approximation to the Navier-Stokes problem.

Let Ω be a bounded domain in $R^d (d = 2, 3)$ assumed to have a Lipschitz-continuous boundary $\partial\Omega$. We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega), W^{s,p}(\Omega)^d$ and their associated norms and seminorms, see e.g. [1,4]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega), H^s(\Omega)^d = W^{s,2}(\Omega)^d$ and $H_0^1(\Omega) = \{v \in H^1(\Omega); v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, $\|\cdot\|_{s,\Omega} = \|_{s,2,\Omega}$ and $|\cdot|_{s,\Omega} = |\cdot|_{s,2,\Omega}$. In some places of this paper, $\|\cdot\|_{2,\Omega}$ should be viewed as piecewise defined if it is necessary. The space $H^{-1}(\Omega)^n$, the dual of $H_0^1(\Omega)^d, d = 1, 2, 3$, will also be used. For $D \subset G \subset \Omega$, we use the notation $D \subset\subset G$ to mean that $\mathrm{dist}(\partial D \setminus \partial\Omega, \partial G \setminus \partial\Omega) > 0$.

Throughout this paper, we shall use the letter c (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences.

2.1. Mixed Finite Element Spaces

For generality, we will not concentrate on any specific mixed finite element space, rather we shall study a class of mixed finite element spaces that satisfy certain assumptions. We shall now describe such assumptions.

Assume that $T^h(\Omega) = \{\tau\}$ is a mesh of Ω with mesh-size function $h(x)$ whose value is the diameter h_τ of the element τ containing x. One basic assumption on the mesh is that it is not exceedingly over-refined locally, namely

*A*0**.** There exists $\gamma \geq 1$ such that

$$h_\Omega^\gamma \leq ch(x) \quad \forall x \in \Omega, \tag{2.1}$$

where $h_\Omega = \max_{x\in\Omega} h(x)$ is the largest mesh size of $T^h(\Omega)$.

This is apparently a very mild assumption and most practical meshes should satisfy this assumption. Sometimes, we will drop the subscript in h_Ω to h for the mesh size on a domain that is clear from the context.

Associated with a mesh $T^h(\Omega)$, let $X_h(\Omega) \subset H^1(\Omega)^d, M_h(\Omega) \subset L^2(\Omega)$ be two finite element subspaces on Ω and set

$$X_h^0(\Omega) = X_h(\Omega) \cap H_0^1(\Omega)^d, \quad M_h^0(\Omega) = M_h(\Omega) \cap L_0^2(\Omega),$$

where

$$L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q dx = 0\}.$$

Given $G \subset \Omega_0 \subset \Omega$, we define $(X_h(G), M_h(G))$ and $T^h(G)$ to be the restriction of $(X_h(\Omega), M_h(\Omega))$ and $T^h(\Omega)$ to G, and

$$X_0^h(G) = \{v \in X_h^0(\Omega); \mathrm{supp}\, v \subset\subset G\}, \quad M_0^h(G) = \{q \in M_h(\Omega); \mathrm{supp}\, q \subset\subset G\}.$$

We now state our basic assumptions on the mixed finite element spaces.