Journal of Computational Mathematics, Vol.24, No.2, 2006, 157–168.

ON THE CONVERGENCE OF THE NONNESTED V-CYCLE MULTIGRID METHOD FOR NONSYMMETRIC AND INDEFINITE SECOND-ORDER ELLIPTIC PROBLEMS *

Huo-yuan Duan and Qun Lin
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

This paper provides a proof for the uniform convergence rate (independently of the number of mesh levels) for the nonnested V-cycle multigrid method for nonsymmetric and indefinite second-order elliptic problems.

Mathematics subject classifications: 65N30.
Key words: Nonnested V-cycle multigrid method, Second-order elliptic problems.

1. Introduction

In this paper we study the convergence of the nonnested V-cycle multigrid method, cf. [2,3]. The nonnestedness is usually caused either by the nature of a specific element (e.g., nonconforming finite elements) or by the nonnested mesh refinement. Due to the varieties of the elements and the triangulations for various problems, nonnestedness is universal, cf. [2], [5], [3].

It is well-known that a general proof of the uniform convergence of the nonnested V-cycle multigrid method had been open for many years, although there have been numerous numerical experiments showing that a uniform convergence rate does exist, see [7], [14], [5], [18], [9] and references cited. Among others, the analysis of the V-cycle for the non-conforming finite element method for the second-order elliptic problem has been and is still an active research subject. Let us mention some works in this aspect. The authors of [14][21] proposed a so-called Galerkin V-cycle nested multigrid method and obtained a uniform convergence rate. Since the iterated intergrid transfer operator is employed and different discrete equations on different levels are solved, when dealing with anisotropic problems, the computational work is huge for this Galerkin V-cycle. Recently, the author of [23] gave a proof under a less regularity requirement for the nonconforming V-cycle of the symmetric and positive definite second-order elliptic problem. Nevertheless, it is not clear if the analysis therein could be carried over to other cases where the nonnestedness may be caused by bubble functions (the bubbles are either artificial or come from cubic and above finite elements) or by unstructured mesh refinements, due to its lengthy analysis and its long list of assumptions. Other related works may be referred to [8], [5], [9], [2], [24], [25], [27], [28], [29], [26]. More importantly, however, *up to now there is no a general convergence proof for the nonnested V-cycle for nonsymmetric and indefinite second-order elliptic problems.*

* Received March 5, 2003; Final revised August 16, 2005.

In this paper, inspired by an argument developed in [4], we give a general convergence proof for the nonnested V-cycle for nonsymmetric and indefinite second-order elliptic problems. Our proof covers all existing nonnested V-cycle where Assumptions A1) and A2) hold (see Section 3 of this paper), including the non-conforming V-cycle with nested meshes [6,10,15], conforming V-cycle with nonnested meshes [9] and Mortar element V-cycle [26, 29, 28, 20]. We obtain a uniform convergence rate (independently of the number of mesh levels), under the condition that the number of pre and post-smoothing steps are sufficiently large and that the coarsest mesh-size is sufficiently small (see Theorem 3.1 and Remark 3.1 of this paper). We point out that, for all existing nonnested V-cycle methods, Assumptions A1) and A2) are valid, see the comments on various nonnested V-cycles in Remark 3.2 of this paper. The key Assumption A1) is the usual regularity-approximation property as in [4,2,3,1], whose verification here requires the full elliptic regularity assumption. The Assumption A2) concerns the approximation property of the coarse-to-fine intergrid transfer operator, which is usually either the interpolation or the L^2 projection operator. Also, the assumption A2) holds for all existing nonnested V-cycles, see Remark 3.2 of this paper. We would like also to remark that it is not clear if our approach could be applied to the case of less elliptic regularity, see related works [27] for nonconforming W-cycle and [23] for nonconforming V-cycle for symmetric and positive definite second-order elliptic problems.

The outline of this paper is as follows. In section 2, we review the nonsymmetric and indefinite second-order elliptic problem and the V-cycle multigrid method as well as some notations. In section 3, we obtain the convergence rate for the nonnested V-cycle multigrid method for nonsymmetric and indefinite second-order elliptic problems.

2. Preliminaries

2.1. Nonsymmetric and indefinite second-order elliptic problem

Let Ω be a bounded, connected domain in $\mathbb{R}^n$, $(n=2,3)$, with Lipschitz continuous boundary $\partial\Omega$. We will use Sobolev spaces $H^k(\Omega)$, with norm $||\cdot||_{H^k(\Omega)}$ and seminorm $|\cdot|_{H^k(\Omega)}$, and $H_0^1(\Omega)=\{v\in H^1(\Omega); v|_{\partial\Omega}=0\}$. We denote by $(\cdot,\cdot)$ the inner product of $L^2(\Omega)(\equiv H^0(\Omega))$ or $(L^2(\Omega))^n$.

We consider the nonsymmetric and indefinite second-order elliptic problem:

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{n}b_i(x)\frac{\partial u}{\partial x_i}+d(x)u=f,\quad \text{in }\Omega,\quad u_{|\partial\Omega}=0, \tag{2.1}$$

where $\mathcal{A}(x):=(a_{ij}(x))\in\mathbb{R}^{n\times n}$ is bounded symmetric and uniformly positive definite in the usual sense, and $a_{ij},b_i\in C^1(\bar{\Omega})$ and $d\in C^0(\bar{\Omega})$. The variational problem of (2.1) reads as follows: Find $u\in U:=H_0^1(\Omega)$ such that

$$a(u,v)=(f,v)\quad \forall v\in U, \tag{2.2}$$

where

$$a(u,v):=\tilde{a}(u,v)+b(u,v), \tag{2.3}$$

$$\tilde{a}(u,v):=(\mathcal{A}\bigtriangledown u,\bigtriangledown v)+(u,v), \tag{2.4}$$

$$b(u,v):=(\boldsymbol{b}\cdot\bigtriangledown u,v)+((d-1)u,v), \tag{2.5}$$

with $\boldsymbol{b}:=(b_1,\cdots,b_n)^T$ and $\bigtriangledown u=(\partial u/\partial x_1,\cdots,\partial u/\partial x_n)^T$.