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MODIFIED MORLEY ELEMENT METHOD FOR A FOURTH ORDER ELLIPTIC SINGULAR PERTURBATION PROBLEM *1)

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Abstract

This paper proposes a modified Morley element method for a fourth order elliptic singular perturbation problem. The method also uses Morley element or rectangle Morley element, but linear or bilinear approximation of finite element functions is used in the lower part of the bilinear form. It is shown that the modified method converges uniformly in the perturbation parameter.

Mathematics subject classification: 65N30.
Key words: Morley element, Singular perturbation problem.

1. Introduction

Let Ω be a bounded polygonal domain of R^2. Denote the boundary of Ω by $\partial\Omega$. For $f \in L^2(\Omega)$, we consider the following boundary value problem of fourth order elliptic singular perturbation equation:

$$\begin{cases} \varepsilon^2 \Delta^2 u - \Delta u = f, & \text{in } \Omega, \\ u|_{\partial\Omega} = \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = 0 \end{cases} \tag{1.1}$$

where $\nu = (\nu_1, \nu_2)^\top$ is the unit outer normal to $\partial\Omega$, Δ is the standard Laplacian operator and ε is a real small parameter with $0 < \varepsilon \leq 1$. When $\varepsilon \to 0$ the differential equation formally degenerates to Poisson equation.

To overcome the C^1 difficult, it is prefer to using nonconforming finite element to solve problem (1.1). Since the differential equation degenerates to Poisson equation as $\varepsilon \to 0$, C^0 nonconforming elements seem better to be used. An C^0 nonconforming finite element was proposed in [4], and its uniform convergence in ε was shown.

It is known that Morley element is not C^0 element and it is divergent for Poisson equation (see [6]). When Morley element is applied to solve problem (1.1), it fails when $\varepsilon \to 0$ (see [4]). On the other hand, we have noticed the remark in the end of paper [4]: the best result uniformly in ε seems to be order of $O(h^{1/2})$ for any finite element method to problem (1.1). Here h is the mesh size. Since Morley element has the least number of element degrees of freedom, we prefer to use a method which still uses the degrees of freedom of Morley element to solve problem (1.1).

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In this paper, we will propose a modified Morley element method for problem (1.1). The method also uses Morley element, but the linear approximation of finite element functions is used in the part of the bilinear form corresponding to the second order differential term. The modified method degenerates to the conforming linear element method for Poisson equation when $\varepsilon = 0$, and this is consistent with the degenerate case of problem (1.1). We will show that the modified method converges uniformly in perturbation parameter ε.

The modified rectangle Morley element method is also considered in this paper.

The paper is organized as follows. The rest of this section lists some preliminaries. Section 2 gives the detail descriptions of the modified Morley element method. Section 3 shows the uniform convergence of the method. The last section gives some numerical results.

For nonnegative integer s, $H^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev space, norm and semi-norm respectively. Let $H_0^s(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^s(\Omega)$ with respect to the norm $\|\cdot\|_{s,\Omega}$ and $(\cdot,\cdot)$ denote the inner product of $L^2(\Omega)$. Define

$$a(v,w) = \int_\Omega \sum_{i,j=1}^{2} \frac{\partial^2 v}{\partial x_i \partial x_j} \frac{\partial^2 w}{\partial x_i \partial x_j} \mathrm{d}x, \quad \forall v, w \in H^2(\Omega). \tag{1.2}$$

$$b(v,w) = \int_\Omega \sum_{i=1}^{2} \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \mathrm{d}x, \quad \forall v, w \in H^1(\Omega). \tag{1.3}$$

The weak form of problem (1.1) is: find $u \in H_0^2(\Omega)$ such that

$$\varepsilon^2 a(u,v) + b(u,v) = (f,v), \quad \forall v \in H_0^2(\Omega). \tag{1.4}$$

Let u^0 be the solution of following boundary value problem:

$$\begin{cases} -\Delta u^0 = f, & \text{in } \Omega, \\ u^0|_{\partial\Omega} = 0 \end{cases} \tag{1.5}$$

The following lemma is shown in paper [4].

Lemma 1.1. *If Ω is convex, then there exists a constant C independent of ε such that*

$$|u|_{2,\Omega} + \varepsilon |u|_{3,\Omega} \le C\varepsilon^{-\frac{1}{2}} \|f\|_{0,\Omega} \tag{1.6}$$

$$|u - u^0|_{1,\Omega} \le C\varepsilon^{\frac{1}{2}} \|f\|_{0,\Omega} \tag{1.7}$$

for all $f \in L^2(\Omega)$.

2. Modified Morley Element Method

For a subset $B \subset R^2$ and r a nonnegative integer, let $P_r(B)$ be the space of all polynomials with degree not greater than r.

Morley Element

Given a triangle T, its three vertices is denoted by a_j, $1 \le j \le 3$. The edge of T opposite a_j is denoted by F_j, $1 \le j \le 3$. Denote the measures of T and F_i by $|T|$ and $|F_i|$ respectively. Morley element can be described by (T, P_T, Φ_T) with

1) T is a triangle.

2) $P_T = P_2(T)$.

3) Φ_T is the vector of degrees of freedom whose components are:

$$v(a_j), \ \frac{1}{|F_j|} \int_{F_j} \frac{\partial v}{\partial \nu} \mathrm{d}s, \quad 1 \le j \le 3$$

for $v \in C^1(T)$.