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NUMERICAL SOLUTIONS OF PARABOLIC PROBLEMS ON UNBOUNDED 3-D SPATIAL DOMAIN [*1)]

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Abstract

In this paper, the numerical solutions of heat equation on 3-D unbounded spatial domain are considered. An artificial boundary Γ is introduced to finite the computational domain. On the artificial boundary Γ, the exact boundary condition and a series of approximating boundary conditions are derived, which are called artificial boundary conditions. By the exact or approximating boundary condition on the artificial boundary, the original problem is reduced to an initial-boundary value problem on the bounded computational domain, which is equivalent or approximating to the original problem. The finite difference method and finite element method are used to solve the reduced problems on the finite computational domain. The numerical results demonstrate that the method given in this paper is effective and feasible.

Mathematics subject classification: 35K05, 65M06, 65M60.
Key words: Heat equation, Artificial boundary, Exact boundary conditions, Finite element method.

1. Introduction

Numerical solutions of heat equation on unbounded 3-D spatial domains are considered. This kind of problems originate from the heat transfer, fluid dynamics, astrophysics, finance or other areas of applied mathematics. Because of the unboundedness of the physical domains, how to numerically solve these problems efficiently is real a challenge.

Strain [14] developed a method to solve the parabolic equations on unbounded domains, which combines the fast Gauss transform with an adaptive refinement scheme. This method can solve heat equation with large timesteps, especially for highly nonuniform or discontinuous initial data.

Artificial boundary method [3, 4, 10, 11] is a powerful tool of the numerical solution for the boundary-valued problems on unbounded domains. By introducing an artificial boundary, the domain is divided into two parts, a finite computational domain and an infinite domain. A suitable boundary condition is imposed on the artificial boundary, such that the solution of the problem with the suitable boundary condition on the artificial boundary on the finite computational domain is a good approximation of the original problem.

For the elliptic problems on unbounded domains, there are many approaches to construct artificial boundary condition to solve them [5, 10, 11, 12], but for the parabolic problems on unbounded domains there are only a few results related to the artificial boundary conditions. L. Halpern and J. Rauch[7] proposed a family of artificial boundary conditions for parabolic equations on unbounded domains, which are local in time, and there are many auxiliary functions involved in the artificial boundary conditions. C. J. Zhu and Q. K. Du [15] studied the

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parabolic problem on an unbounded domain by a semi-discrete approach in time. On each time step, they used the nature boundary element method to solve an elliptic problem on unbounded domain. Recently, H. Han and Z. Huang [8, 9] gave the exact boundary conditions for the heat problems on unbounded domains in one dimension and in two dimensions. Furthermore, a series of artificial boundary conditions are given. Han and Zheng [13] derived the nonreflecting boundary conditions for acoustic problems in three dimensions.

In this paper, the exact boundary condition is derived on the given artificial boundary Γ for the parabolic problem on unbounded three dimensional spatial domain, that is, the relationship between $\frac{\partial u}{\partial n}\big|_\Gamma$ and $\frac{\partial u}{\partial t}\big|_\Gamma$ is given. Moreover, a series of artificial boundary conditions with high accuracy are obtained. By the artificial boundary conditions, a family of approximate problems of the original problem on the bounded computational domain are constructed. The stability of the approximate problem is proved. Finally, numerical examples manifest the feasibility and effectiveness of the given method.

2. The Artificial Boundary Condition

Let $D \subset \mathbb{R}^3$ denote a bounded domain, namely $D \subset B(0,a) = \{x \in \mathbb{R}^3 \mid ||x|| \leq a\}$ with $a > 0$. Suppose

$$D^c = \mathbb{R}^3 \backslash \overline{D}, \quad \Omega_c^T = D^c \times (0,T], \quad \Gamma_0 = \partial D \times (0,T].$$

Consider the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} - \triangle u = f(x,t), \qquad (x,t) \in \Omega_c^T \tag{2.1}$$

$$u\Big|_{\Gamma_0} = g(x,t), \qquad (x,t) \in \Gamma_0, \tag{2.2}$$

$$u\Big|_{t=0} = u_0(x), \qquad x \in D^c, \tag{2.3}$$

$$u \to 0, \qquad \text{when } ||x|| \to +\infty, \tag{2.4}$$

where $f(x,t), g(x,t)$ and $u_0(x)$ are given smooth functions and $f(x,t)$, $u_0(x)$ vanish outside the ball $B(0,a)$, namely

$$f(x,t) = 0, \quad u_0(x) = 0, \text{ if } ||x|| \geq a.$$

We introduce an artificial boundary $\Gamma = \{(x,t) \mid ||x|| = b, 0 < t \leq T\}$ with $b > a$ to divide domain Ω_c^T into two parts,

$$\begin{aligned} \Omega_b^T &= \{(x,t) \mid x \in D^c \text{ and } ||x|| < b, 0 < t < T\}, \\ \Omega_e^T &= \{(x,t) \mid ||x|| \geq b, 0 < t \leq T\}. \end{aligned}$$

If we can seek a suitable boundary condition on Γ, problem (2.1)-(2.4) can be reduced to a problem on the bounded computational domain Ω_i^T. In the sphere coordinate, the restriction of the solution $u(r,\theta,\phi,t)$ of problem (2.1)-(2.4) on the unbounded domain Ω_e^T satisfies

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial u}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2}, \quad (r,\theta,\phi,t) \in \Omega_e^T, \tag{2.5}$$

$$u|_{r=b} = u(b,\theta,\phi,t), \tag{2.6}$$

$$u|_{t=0} = 0, \tag{2.7}$$

$$u \to 0, \quad \text{when } \ r \to +\infty, \tag{2.8}$$

where $\Omega_e^T = \{r > b, \theta \in [0,\pi], \phi \in [0, 2\pi], t \in [0,T]\}$.

Since $u(b,\theta,\phi,t)$ is unknown, problem (2.5)-(2.8) is an uncompleted posed problem; it can't be solved independently. If $u(b,\theta,\phi,t)$ is given, problem (2.5)-(2.8) is well posed, so the solution $u(r,\theta,\phi,t)$ of (2.5)-(2.8) can be given by $u(b,\theta,\phi,t)$.