Journal of Computational Mathematics, Vol.23, No.4, 2005, 419–424.

ITERATIVE METHODS FOR THE FORWARD-BACKWARD HEAT EQUATION *1)

Xiao-liang Cheng
(*Department of Mathematics, Zhejiang University, Hangzhou 310028, China*)

Jie Sun
(*Department of Mathematics, Zhejiang University, Hangzhou 310028, China*)
(*Department of Mathematics, Zhejiang University of Finance and Economics, Hangzhou 310012, China*)

Abstract

In this paper we propose the finite difference method for the forward-backward heat equation. We use a coarse-mesh second-order central difference scheme at the middle line mesh points and derive the error estimate. Then we discuss the iterative method based on the domain decomposition for our scheme and derive the bounds for the rates of convergence. Finally we present some numerical experiments to support our analysis.

Mathematics subject classification: 65N22, 65M06, 35K05, 65N55.
Key words: Forward-backward heat equation, Finite difference method, Iterative method, Coarse mesh.

1. Introduction

In this paper, we consider the following boundary value problem of a forward-backward parabolic equation:

$$\begin{cases} a(x)u_t - u_{xx} = f(x,t), \quad (x,t) \in \Omega = (-1,1)\times(0,1), \\ u(x,0) = 0, \quad 0 \le x \le 1, \\ u(x,1) = 0, \quad -1 \le x \le 0, \\ u(1,t) = 0,\ u(-1,t) = 0, \quad 0 < t < 1, \end{cases} \tag{1.1}$$

where $a(x) > 0$ for $x > 0$, $a(x) < 0$ for $x < 0$ and $a(0) = 0$. For example, $a(x) = x$ or $a(x) = x^m$ with m the odd integer. The problem (1.1) arises in a variety of applications such as randomly accelerated particle problem and fluid flow near a boundary where separation occurs, see [1, 2] for the details. So far there are several numerical approach to this problem, for example, the finite difference method[1], least square method[5] and Galerkin finite element method[3, 4, 7, 8].

The purpose of this paper is to present a finite difference scheme to equation (1.1). Unlike the standard way in [1], we use a coarse mesh second-order central difference scheme at the mesh points lie on the middle line $x = 0$, $0 < t < 1$. We prove the error estimates $O(\tau + h^2 + H^3)$ with time mesh size τ and space mesh size h and coarse mesh size H. Then we discuss the iterative method based on the domain decomposition method for our scheme and obtain bounds of the convergent rate with $1 - H$, which is better than that $1 - h$ in [1]. In the last section we present some numerical results to support our analysis.

2. The Difference Scheme

We first specify the grids. Let $h = 1/M$ and $x_i = ih$ for $i = 0, \pm1, \pm2, \cdots, \pm M$. Let $\tau = 1/N$ and $t_j = j\tau$ for $j = 0, 1, \cdots, N$.

* Received February 3, 2004; final revised July 19, 2004.
1) The project was supported by National Science Foundation(Grant No. 10471129).

We use the backward and forward difference scheme on domain $x > 0$ and $x < 0$ respectively and second order central difference scheme on the line $x = 0$ with coarse mesh $H = m_0 h$ for some given positive integer m_0. Denote z_i^j $(-M+1 \le i \le M-1,\ 1 \le j \le N-1)$ is the approximation solution for the exact solution at point $(ih, j\tau)$. Then

$$\begin{cases} a_i \dfrac{z_i^{j+1} - z_i^j}{\tau} - \dfrac{z_{i-1}^{j+1} - 2z_i^{j+1} + z_{i+1}^{j+1}}{h^2} = f_i^{j+1}, \quad 1 \le i \le M-1,\ 0 \le j \le N-2, \\ z_i^0 = 0, \quad 1 \le i \le M-1, \\ z_M^j = 0, \quad 1 \le j \le N-1; \end{cases} \tag{2.1}$$

$$\begin{cases} a_i \dfrac{z_i^{j+1} - z_i^j}{\tau} - \dfrac{z_{i-1}^{j} - 2z_i^{j} + z_{i+1}^{j}}{h^2} = f_i^{j}, \quad -M+1 \le i \le -1,\ 1 \le j \le N-1, \\ z_i^N = 0, \quad -M+1 \le i \le -1, \\ z_{-M}^j = 0, \quad 1 \le j \le N-1; \end{cases} \tag{2.2}$$

and

$$-\frac{z_{m_0}^j - 2z_0^j + z_{-m_0}^j}{H^2} = f_0^j, \quad 1 \le j \le N-1. \tag{2.3}$$

Here $a_i = a(ih)$ and $f_i^j = f(ih, j\tau)$. For $m_0 = 1$, it is the same method proposed in [1].

It is convenient to introduce the set of all mesh points by $\mathcal{N}$,

$$\mathcal{N} = \{(ih, j\tau) \,|\, -M+1 \le i \le M-1,\ 1 \le j \le N-1\}.$$

Further we split $\mathcal{N} = \mathcal{N}_w \cup \mathcal{N}_v \cup \mathcal{N}_\psi$ into three disjoint subsets as follows,

$$\mathcal{N}_w = \{(ih, j\tau) \,|\, 1 \le i \le M-1,\ 1 \le j \le N-1\},$$

$$\mathcal{N}_v = \{(ih, j\tau) \,|\, -M+1 \le i \le -1,\ 1 \le j \le N-1\},$$

$$\mathcal{N}_\psi = \{(0, j\tau) \,|\, 1 \le j \le N-1\}.$$

We write the linear system (2.1)-(2.3) in the matrix form $\mathcal{P}Z = F$ with

$$\mathcal{P} = \begin{pmatrix} A_{vv} & 0 & A_{v\psi} \\ 0 & A_{ww} & A_{w\psi} \\ A_{v\psi} & A_{w\psi} & A_{\psi\psi} \end{pmatrix} \tag{2.4}$$

The vector $Z = (Z_v, Z_w, Z_\psi)^T$ with

$$Z_v = (z_{-M+1}^1, z_{-M+1}^2, \cdots, z_{-1}^1, \cdots, z_{-1}^{N-1})^T,$$

$$Z_w = (z_1^1, z_1^2, \cdots, z_{M-1}^1, \cdots, z_{M-1}^{N-1})^T,$$

and $Z_\psi = (z_0^1, z_0^2, \cdots, z_0^{N-1})^T$. And F the vector defined on the mesh points $\mathcal{N}$ of the function $f(x,t)$.

Let u be the exact solution of problem (1.1). Denote the error

$$E_i^j = u(ih, j\tau) - z_i^j, \quad (ih, j\tau) \in \mathcal{N}.$$

We use the maximum norm

$$\|E\|_{\mathcal{N}} = \max_{(ih,j\tau)\in\mathcal{N}} |E_i^j|.$$

Now we will prove the following error estimates.

Theorem 2.1. *Suppose that $\frac{1}{2}|\partial^2 u/\partial t^2|$ and $\frac{1}{12}|\partial^4 u/\partial x^4|$ are bounded by constant C_0 on $\bar{\Omega}$, the closure of Ω. Then*

$$\|E\|_{\mathcal{N}} \le \frac{1}{2} C_0 (\tau + h^2 + H^3). \tag{2.5}$$