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STABILITY ANALYSIS OF RUNGE-KUTTA METHODS FOR NONLINEAR SYSTEMS OF PANTOGRAPH EQUATIONS [*1)]

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Abstract

This paper is concerned with numerical stability of nonlinear systems of pantograph equations. Numerical methods based on $(k,l)-$algebraically stable Runge-Kutta methods are suggested. Global and asymptotic stability conditions for the presented methods are derived.

Mathematics subject classification: 65L06, 65L20.
Key words: Nonlinear pantograph equations, Runge-Kutta methods, Numerical stability, Asymptotic stability.

1. Introduction

Consider the following systems of the pantograph equations

$$\begin{cases} y'(t) = f(t, y(t), y(pt)), & t > 0, \\ y(0) = \eta, & \eta \in C^N, \end{cases} \tag{1.1}$$

where $f : [0, +\infty) \times C^N \times C^N \to C^N$ is a given function and $p \in (0, 1)$ is a real constant. For applications of the systems(1.1), we refer to Iserles[1].

In order to investigate the stability of numerical methods for the pantograph equations, the scalar linear pantograph equations

$$y'(t) = \lambda y(t) + \mu y(pt),$$

where $\lambda, \mu \in C$ and $p \in (0, 1)$ are constants, have been used as the test problem and many significant results have been derived(cf.[2-10, 16, 17]). However, little attention has been paid to the nonlinear case of the form (1.1). In 2002, Zhang and Sun[11] considered nonlinear stability of one-leg $\theta-$methods for (1.1) and obtained some results of global and asymptotic stability. On the basis of their works, the present paper further deal with numerical stability of (k,l)-algebraically stable Runge-Kutta methods with variable stepsize (introduced by Liu[9]) for the nonlinear systems (1.1). Global and asymptotic stability conditions for the presented methods are derived.

2. Runge-Kutta Methods with Variable Stepsize

In this section, we consider the adaptation of Runge-Kutta methods for solving (1.1). Let (A, b, c) denotes a given Runge-Kutta method with matrix $A = (a_{ij}) \in R^{s\times s}$ and vectors $b = (b_1, b_2, \ldots, b_s)^T \in R^s$, $c = (c_1, c_2, \ldots, c_s)^T \in R^s$. In this paper, we always assume that $c_i \in [0, 1]$, $i = 1, 2, \ldots, s$. The application of the Runge-Kutta method (A, b, c) to (1.1) yields

$$\begin{cases} Y_i^{(n)} = y_n + h_{n+1} \sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j^{(n)}, \widetilde{Y}_j^{(n)}), & i = 1, 2, \ldots, s, \\ y_{n+1} = y_n + h_{n+1} \sum\limits_{i=1}^{s} b_i f(t_n + c_i h, Y_i^{(n)}, \widetilde{Y}_i^{(n)}), & n = 0, 1, 2, \ldots, \end{cases} \tag{2.1}$$

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where $h_{n+1} = t_{n+1} - t_n$, y_n, $Y_i^{(n)}$ and $\widetilde{Y}_i^{(n)}(n \geq 0, i = 1, 2, \ldots, s)$ are approximations to $y(t_n)$, $y(t_n + c_i h_{n+1})$ and $y(p(t_n + c_i h_{n+1}))$ respectively.

Since a serious storage problem is created when the computation for (1.1) with constant stepsize is run on any computer, we consider a variable stepsize strategy introduced by Liu[9] and Bellen et al.[2] to resolve the storage problem. The grid points are selected as follows(cf. [11]).

First, divide $[0, +\infty)$ into a set of infinite bounded intervals, that is

$$[0, +\infty) = \bigcup_{l=0}^{\infty} D_l,$$

where $D_0 = [0, \gamma]$ with a given positive number γ and $D_l = (T_{l-1}, T_l](l \geq 1)$ with $T_l = p^{-l}\gamma$. Then, partition every primary interval $D_l(l \geq 1)$ into equal m subintervals. Thus the grid points on $[0, +\infty)/D_0$ are determined by

$$t_n = T_{\lfloor (n-1)/m \rfloor} + (n - \lfloor (n-1)/m \rfloor m)h_n, \qquad n \geq 1,$$

where $\lfloor x \rfloor$ denotes the maximal integer which not exceeds x. On D_0, choose $t_0 = \gamma$, $t_{-(m+1)} = 0$, $t_{-i} = pt_{m-i}$, $i = m, m-1, \ldots, 1$, as grid points. The corresponding numerical solutions y_0, y_{-i} and $Y_j^{(-i)}(i = m+1, m, \ldots, 1,\ j = 1, 2, \ldots, s)$ are assumed to exist. So the function $\varphi(t) := pt$ has these properties:

$$\begin{aligned} &[S1] \quad \varphi(t_n) = t_{n-m}, && n \geq 0, \\ &[S2] \quad \varphi(D_{n+1}) = D_n, && n \geq 1, \\ &[S3] \quad \varphi(h_n) = h_{n-m}, && n \geq 1, \end{aligned}$$

and the stepsize sequence $\{h_n\}$ is determined by

$$h_n = \begin{cases} p\gamma, & n = -m, \\ \frac{(1-p)\gamma}{m}, & n = -m+1, -m+2, \ldots, -1, 0, \\ \frac{(1-p)\gamma}{mp^{\lfloor (n-1)/m \rfloor + 1}}, & n = 1, 2, 3, \ldots. \end{cases} \tag{2.2}$$

Properties [S1]-[S3] imply that the choice of grid points has removed the computational storage problem for (1.1) and the method (2.1) can be written as

$$\begin{cases} Y_i^{(n)} = y_n + h_{n+1} \sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j^{(n)}, Y_j^{(n-m)}), & i = 1, 2, \ldots, s, \\ y_{n+1} = y_n + h_{n+1} \sum\limits_{i=1}^{s} b_i f(t_n + c_i h, Y_i^{(n)}, Y_i^{(n-m)}), & n = 0, 1, 2, \ldots, \end{cases} \tag{2.3}$$

3. Stability Analysis of the Methods

In order to study the stability of the methods (2.3), consider the perturbed systems of (1.1)

$$\begin{cases} z'(t) = f(t, z(t), z(pt)), & t > 0, \\ z(0) = \varsigma, & \varsigma \in C^N, \end{cases} \tag{3.1}$$

Similarly, applying method (2.3) to the systems (3.1) yields

$$\begin{cases} Z_i^{(n)} = z_n + h_{n+1} \sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h, Z_j^{(n)}, Z_j^{(n-m)}), & i = 1, 2, \ldots, s, \\ z_{n+1} = z_n + h_{n+1} \sum\limits_{i=1}^{s} b_i f(t_n + c_i h, Z_i^{(n)}, Z_i^{(n-m)}), & n = 0, 1, 2, \ldots, \end{cases} \tag{3.2}$$

where z_n and $Z_i^{(n)}$ are approximations to $z(t_n)$ and $z(t_n + c_i h_{n+1})$ respectively.

Both (1.1) and (3.1), we assume that the function f satisfies

$$\begin{cases} Re\langle u_1 - u_2, f(t, u_1, v) - f(t, u_2, v)\rangle \leq \alpha \|u_1 - u_2\|^2, & t > 0, \quad u_1, u_2, v \in C^N, \\ \|f(t, u, v_1) - f(t, u, v_2)\| \leq \beta \|v_1 - v_2\|, & t > 0, \quad u, v_1, v_2 \in C^N, \end{cases} \tag{3.3}$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote a given inner product and the corresponding norm in complex $N-$dimensional space C^N respectively. In the following, all systems (1.1) with (3.3) will be