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A REVISED CONJUGATE GRADIENT PROJECTION ALGORITHM FOR INEQUALITY CONSTRAINED OPTIMIZATIONS *

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Abstract

A revised conjugate gradient projection method for nonlinear inequality constrained optimization problems is proposed in the paper, since the search direction is the combination of the conjugate projection gradient and the quasi-Newton direction. It has two merits. The one is that the amount of computation is lower because the gradient matrix only needs to be computed one time at each iteration. The other is that the algorithm is of global convergence and locally superlinear convergence without strict complementary condition under some mild assumptions. In addition the search direction is explicit.

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1. Introduction

Consider the optimization problem

$$\min\{f(x) : g_j(x) \leq 0,\ j \in I,\ x \in R^n\}, \tag{1}$$

where $f(x),\ g_j(x) : R^n \to R,\ j \in I = \{1, 2, \ldots, m\}$.

We know that the quasi-Newton method is one of the most effective methods for solving nonlinear optimal problems because of the property of superlinear convergence. Some people investigated the variable metric algorithms for constrained optimization problems, such as [4, 5, 7, 8, 9, 13]. At present, the research on this topic is still active due to various improvements both in theory and applications. It is one of important results that the search direction of the method is constructed by combining the conjugate projective gradient with the quasi-Newton direction. However, the assumption of strict complementary condition, which is very strong, is necessary for keeping the superlinear convergence. Bonnans and Launay [1] proposed a globally and superlinearly convergent method without strict complementary condition. But it needs sufficiently curvature condition and needs to solve two quadratic sub-programmings in each iteration. Generally, the search directions of constrained quasi-Newton methods are composed of two different approaches. In fact, the search direction is determined by quasi-Newton direction under some conditions and determined by the gradient projection direction under

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other conditions in the same iteration[12, 14]. This leads to the great amount of computation generally.

In order to overcome the defects in the stated methods above—strong assumption and great amount of computation, we present a revised conjugate projective gradient method. By means of schemes of ϵ-active constraint set and the explicit conjugate projection gradient direction with respect to a positive definite matrix, our algorithm is more harmonious and effective. There are two merits in our method. The first is that the algorithm is of global convergence and locally superlinear convergence without strict complementary condition. The second is that we only need to compute the conjugate projection matrix one time to obtain two search directions at each iteration, so the method presented greatly decreases the amount of computation.

This paper is organized as follow. In Section 2, we present the algorithm. The convergence of the algorithm is discussed in Section 3. In Section 4, we analyze the rate of convergence of the algorithm. In the last section, we show the numerical tests with four examples.

2. Algorithm

Denote $X = \{x : g_j(x) \le 0\}$, and $J(x) = \{j \in I : g_j(x) \ge -\varepsilon\}$. We need following two assumptions in the paper.

A1. *$f(x)$ and $g_j(x)$ are continuously differentiable for any $j \in I$.*

A2. *$\{\nabla g_j(x),\ j \in J(x)\}$ is linearly independent for $x \in X$.*

Let us introduce some notes. At current point x_k, we define

$$A_k = (\nabla g_j(x_k),\ j \in J(x_k)),$$
$$g_{J_k} = (g_j(x_k),\ j \in J(x_k))^T.$$

The conjugate projection w.r.t. a given symmetric positive definite matrix H_k is

$$P_k = H_k - H_k A_k B_k \tag{2}$$

where $B_k = (A_k^T H_k A_k)^{-1} A_k^T H_k$.

We define

$$d_k^0 = -P_k \nabla f(x_k) - B_k^T g_{J_k}, \tag{3}$$

and

$$\lambda_k = -B_k \nabla f(x_k) + (A_k^T H_k A_k)^{-1} g_{J_k} = \lambda_k^1 + \lambda_k^2, \tag{4}$$

where, $\lambda_k^1 = -B_k \nabla f(x_k)$, $\lambda_k^2 = (A_k^T H_k A_k)^{-1} g_{J_k}$.

If the set $J(x_k) = \emptyset$, the algorithm is an ordinary quasi-Newton method. In the following, we always assume that $J(x_k) \neq \emptyset$.

To simplify, we denote $x_k, P_k, A_k, \cdots$ as $x, P, A, \cdots$, and J stands for $J(x)$.

Lemma 1. *P is a positive semi-definite matrix, and $PA = 0$, $BA = E$, where E is a $|J| \times |J|$ identity matrix.*

Theorem 1. *If $x \in X, d^0 = 0, \lambda \ge 0$, then x is a KKT point of problem* (1).

Proof. From $d^0 = 0$, we have

$$\begin{aligned} 0 &= -H\nabla f(x) + HA(A^T HA)^{-1} A^T H \nabla f(x) - HA(A^T HA)^{-1} g_J \\ &= -H\nabla f(x) - HA\lambda \end{aligned}$$

and

$$0 = A^T d^0 = -A^T P \nabla f(x) - (BA)^T g_J = -g_J.$$

Hence, there exists $\lambda \ge 0$ such that $\nabla f(x) + A\lambda = 0$ and $\lambda_j g_j = 0,\ j \in J$.

Now state the algorithm as follows.