Journal of Computational Mathematics, Vol.23, No.1, 2005, 1–16.

A SECOND ORDER CONTROL-VOLUME FINITE-ELEMENT LEAST-SQUARES STRATEGY FOR SIMULATING DIFFUSION IN STRONGLY ANISOTROPIC MEDIA *1)

Jayantha Pasdunkorale A.
(*Department of Mathematics, University of Ruhuna, Matara, Sri Lanka 81000*)

Ian W. Turner
(*School of Mathematical Sciences, Queensland University of Technology, Brisbane, Australia 4001*)

Abstract

An unstructured mesh finite volume discretisation method for simulating diffusion in anisotropic media in two-dimensional space is discussed. This technique is considered as an extension of the fully implicit hybrid control-volume finite-element method and it retains the local continuity of the flux at the control volume faces. A least squares function reconstruction technique together with a new flux decomposition strategy is used to obtain an accurate flux approximation at the control volume face, ensuring that the overall accuracy of the spatial discretisation maintains second order. This paper highlights that the new technique coincides with the traditional shape function technique when the correction term is neglected and that it significantly increases the accuracy of the previous linear scheme on coarse meshes when applied to media that exhibit very strong to extreme anisotropy ratios. It is concluded that the method can be used on both regular and irregular meshes, and appears independent of the mesh quality.

Mathematics subject classification: 74S10, 76M12, 74S05, 76M10.
Key words: Error correction term, Shape Functions, Gradient Reconstruction, Flux Approximation.

1. Introduction

An accurate approximation of the flux at the control volume face is one of the challenges in finite volume discretisation techniques [1, 2] for simulating transport in highly anisotropic media on arbitrary shaped meshes [3, 4, 5, 6, 7, 8, 9]. In the past the hybrid control-volume finite-element method has been used to approximate the necessary fluxes [10, 11, 12, 13], where it has been shown that the use of very fine meshes produces accurate results, however the computational cost is very high, especially for problems in three dimensions [10]. On the other hand this technique fails to provide accurate results on coarse meshes for strongly orthotropic media [13].

This work builds upon the finite volume flux decomposition technique proposed in [9, 14] and seeks to resolve the problems associated with using that scheme under extreme anisotropy ratios, whereby divergence was observed in the iterative solution of the underlying linear system. It was concluded in that work that the main difficulty arose due to the explicit treatment of the cross-diffusion component of the flux, which in some instances of strong anisotropy carries with it the most important contribution of the entire flux. The key factor in resolving this issue is to recover a proportion of this cross-diffusion term in an implicit manner.

* Received April 8, 2003.
1) The first author was supported by the Queensland University of Technology under the IPRS scholarship programme and the leave granted for the studies from the University of Ruhuna, Sri Lanka.

One notes that the hybrid (control volume finite element - CVFE) scheme [9, 10, 12, 13] naturally treats the implicitness of the cross-diffusion term via use of the shape functions. Furthermore, this scheme has performed accurately and efficiently (in terms of overall computational overheads) for a variety of isotropic diffusion problems and diffusion problems involving relatively small anisotropy ratios. Unfortunately, the CVFE scheme fails to provide accurate results on coarse meshes when it is used to simulate diffusion in media that exhibit strong to extreme anisotropy ratios. This downfall of CVFE method provides the motivation for this research. It seems reasonable to try improving the accuracy of this scheme using some of the innovative ideas proposed in [9], especially since the hybrid method is straightforward to implement and finds application in a wide range of problems resolved using finite-volume finite-element paradigms. In this work a hybrid scheme is derived that uses a weighted least squares function reconstruction technique to increases the linear accuracy of the scheme summarised in [9] to second order accuracy. The overall appearance of the new hybrid scheme retains the previous linear shape function component of the flux term and includes an explicitly treated correction term that utilises locally estimated derivatives to improve the order of the flux approximation. The attraction of the new scheme is twofold. Firstly, if the correction term is neglected the scheme is identical to the previously proposed linear hybrid scheme, thus enabling it to be accommodated easily into existing codes. Secondly, the cost of estimating the correction term is not overly demanding in terms of computational cost since most of the required terms utilize matrices whose coefficients involve only geometrical mesh properties, which can be decomposed and stored during the initialisation of the code and used thereafter during processing. Most importantly, the new scheme provides the appropriate amount of implicitness to the cross-diffusional component of the flux to overcome the problems reported in [14] for extreme anisotropy ratios. The new scheme works well and has provided accurate simulation results for a wide range of benchmark problems tested, the most important of which are reported here in Section 3.

For anisotropic transport problems, the flux term is given by $\mathbf{q} = -K\nabla\phi$ where $K = \begin{pmatrix} k_{xx} & k_{xy} \\ k_{yx} & k_{yy} \end{pmatrix}$. This situation arises in problems such as heat and mass transfer during drying processes, groundwater flow, atmospheric dispersion, heat conduction in solar power collector plates, microwave and convective heating of hygroscopic materials, thermo-elastic stresses and displacements of anisotropic materials, and manufacturing of composite materials [10, 15, 16, 17, 18, 19, 20, 21].

In this work the following two dimensional unsteady anisotropic diffusion equation for a finite rectangular domain $\Omega = [0, L] \times [0, M]$ is considered.

$$\frac{\partial\psi}{\partial t} - \nabla.(K\nabla\phi) = 0 \quad \text{on} \quad \Omega \quad \text{for} \quad 0 < t \leq T < \infty \tag{1}$$

where ψ is a function of ϕ. The boundary conditions and initial condition are defined as follows:

$$-(K\nabla\phi).\mathbf{n}_b = h(\phi - \phi_s) \quad \text{on} \quad \partial\Omega \quad \text{for} \quad 0 < t \leq T < \infty$$

$$\phi(x, y, 0) = F(x, y) \quad \text{in} \quad \Omega$$

where $\mathbf{n}_b$ is the outward unit normal vector at the boundary $\partial\Omega$ and ϕ_s is a constant associated with the boundary conditions.

2. Finite Volume Discretisation

The finite volume discretisation [1, 2, 13] of the diffusion Eq. (1) over the control volume