Journal of Computational Mathematics, Vol.23, No.1, 2005, 17–26.

ON SOLUTIONS OF MATRIX EQUATION $AXA^T + BYB^T = C$ *1)

Yuan-bei Deng
(*ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)
(*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*)

Xi-yan Hu
(*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China*)

Abstract

By making use of the quotient singular value decomposition (QSVD) of a matrix pair, this paper establishes the necessary and sufficient conditions for the existence of and the expressions for the general solutions of the linear matrix equation $AXA^T + BYB^T = C$ with the unknown X and Y, which may be both symmetric, skew-symmetric, nonnegative definite , positive definite or some cross combinations respectively. Also, the solutions of some optimal problems are derived.

Mathematics subject classification: 15A24, 65F53.
Key words: Matrix equation, Matrix norm, QSVD, Constrained condition, Optimal problem.

1. Introduction

It has been of interest for many authors to solve the linear matrix equations under constrained conditions. For the cases of one unknown matrix, such as $AX = B$ or $AXB = C$, the discussions can be seen in literatures [6,11,12,14,16] and [17]. The authors in [2,3,5,13] and [15] considered the solutions of the following linear matrix equation with two unknown matrices

$$AXB + CYD = E, \tag{1.1}$$

which originates from the applications to output feedback pole assignment problems in control theory and from an inverse scattering problem. As special cases, Jameson and Kreindler (1973), Jameson, Kreindler and Lancaster(1992), and Dobovisek (2001) developed the consistent conditions and representations of the solutions of homogeneous equations

$$AX \pm YB = 0 \tag{1.2}$$

with X or Y symmetric(Hermitian), nonnegative definite or positive definite and some cross combinations respectively.

In this paper, we discuss the symmetric matrix equation

$$AXA^T + BYB^T = C \tag{1.3}$$

with the unknown X and Y both symmetric, skew-symmetric, nonnegative definite , positive definite or some cross combinations respectively, which has been studied in [3] for the case X and Y are both symmetric by using the general singular value decomposition (GSVD).

* Received January 7, 2003; final revised July 17, 2004.
1) Supported by the $NNSF$ of China.

Let $R^{m\times n}$ denote the set of all real $m\times n$ matrices, $SR^{n\times n}$, $AR^{n\times n}$, $SR_0^{n\times n}$, $SR_+^{n\times n}$ and $OR^{n\times n}$ are the sets of all real symmetric , skew-symmetric, symmetric nonnegative definite, symmetric positive definite and orthogonal $n\times n$ matrices respectively. When the size is clear, we also write $A\geq 0$ or $A>0$ to denote that A is symmetric nonnegative definite or symmetric positive definite matrix, and $A\geq B(A>B)$ means $A-B\geq 0(A-B>0)$. For $A\in R^{m\times n}$, let A^T, A^+ and $R(A)$ be, respectively, the transpose, the Moore-Penrose inverse and the column space of A. $\|\cdot\|_F$ stands for the Frobenius norm of a matrix, $A*B$ represents the Hadamard product of A and B.

Let $A\in R^{m\times n}, B\in R^{m\times p}$, $C\in R^{m\times m}$, $S_1=SR^{n\times n}$, $S_2=SR_0^{n\times n}$,$S_3=SR_+^{n\times n}$, $S_4=AR^{n\times n}$, $T_1=SR^{p\times p}$, $T_2=SR_0^{p\times p}$,$T_3=SR_+^{p\times p}$, $T_4=AR^{p\times p}$, in the next sections the following problems are considered.

Problem I. Given A, B and C, and let

$$L_{ij}=\{[X,Y]: X\in S_i, Y\in T_j, AXA^T+BYB^T=C\}, \tag{1.4}$$

find the consistent conditions for $L_{ij}\neq\emptyset$, and if the conditions hold, find the expression of $[X,Y]\in L_{ij}$.

problem II. Find $[\hat{X},\hat{Y}]\in L_{ii}$, such that

$$\left\|[\hat{X},\hat{Y}]\right\|_F=\left[\|\hat{X}\|_F^2+\|\hat{Y}\|_F^2\right]^{\frac{1}{2}}=\min. \tag{1.5}$$

This paper is organized as follows. In section 2, we introduce some preliminaries and give the solutions of Problem I and Problem II on L_{11}. In section 3, we establish the solutions of Problem I on L_{12} and L_{13} . In section 4, we provide the solutions of Problem I and Problem II on L_{22}, the solutions of Problem I on L_{23} and L_{33}. Finally in section 5, we discuss the solutions of Problem I and Problem II on L_{44}.

2. Preliminaries and the Solution on L_{11}

We first introduce two lemmas about nonnegative definite and positive definite matrices, see [1], [8] and [18, p325].

Lemma 2.1. *Given matrix* $H=\begin{pmatrix} E & F\\ F^T & G\end{pmatrix}$ *with* $E\in R^{n_1\times n_1}$, $F\in R^{n_1\times n_2}$, $G\in R^{n_2\times n_2}$, *then the following statements are equivalent.*

(i) $H\geq 0$;

(ii) $E\geq 0, G-F^TE^+F\geq 0$ *and* $R(F)\subseteq R(E)$;

(iii) $G\geq 0, E-FG^+F^T\geq 0$ *and* $R(F^T)\subseteq R(G)$.

Lemma 2.2. *Given matrix* $H=\begin{pmatrix} E & F\\ F^T & G\end{pmatrix}$ *with* $E\in R^{n_1\times n_1}$, $F\in R^{n_1\times n_2}$,$G\in R^{n_2\times n_2}$, *then the following statements are equivalent.*

(i) $H>0$;

(ii) $E>0, G-F^TE^{-1}F>0$;

(iii) $G>0, E-FG^{-1}F^T>0$.

The quotient singular value decomposition (QSVD) of a matrix pair $[A,B]$ is stated as follows(cf. [4]), compared with the GSVD, it has a simple form.

Lemma 2.3. *Given two matrices* $A\in R^{m\times n}, B\in R^{m\times p}$, *the QSVD of* $[A,B]$ *is*

$$A=M{\sum}_A U^T, \quad B=M{\sum}_B V^T, \tag{2.1}$$