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HAT AVERAGE MULTIRESOLUTION WITH ERROR CONTROL IN 2-D *1)

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Abstract

Multiresolution representations of data are a powerful tool in data compression. For a proper adaptation to the singularities, it is crucial to develop nonlinear methods which are not based on tensor product. The hat average framework permets develop adapted schemes for all types of singularities. In contrast with the wavelet framework these representations cannot be considered as a change of basis, and the stability theory requires different considerations. In this paper, non separable two-dimensional hat average multiresolution processing algorithms that ensure stability are introduced. Explicit error bounds are presented.

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Key words: Stability, Hat average, Multiresolution, Non separable.

1. Introduction

Multiresolution representations are one of the most efficient tools for data compression. The multi-scale representation of a signal is well adapted to *quantization* or simple *thresholding.*

A discrete sequence f^L is encoded to produce a multi-scale representation of its information contents, $(f^0, \bar{e}^1, \bar{e}^2, \dots, \bar{e}^L)$; this representation is then processed and the end result of this step is a modified multi-scale representation $(\hat{f}^0, \hat{e}^1, \hat{e}^2, \dots, \hat{e}^L)$ which is *close* to the original one, i.e. such that (in some norm)

$$||\hat{f}^0 - f^0|| \leq \epsilon_0 \qquad ||\hat{e}^k - \bar{e}^k|| \leq \epsilon_k \quad 1 \leq k \leq L,$$

where the truncation parameters $\epsilon_0, \epsilon_1, \dots, \epsilon_L$ are chosen according to some criteria specified by the user. After decoding the processed representation, we obtain a discrete set $\hat{f}^L$ which is expected to be *close* to the original discrete set f^L. Thus, some form of stability is needed, i.e. we must require that

$$||\hat{f}^L - f^L|| \leq \sigma(\epsilon_0, \epsilon_1, \dots, \epsilon_L)$$

where $\sigma(\cdot, \dots, \cdot)$ satisfies

$$\lim_{\epsilon_l \to 0,\ 0 \leq l \leq L} \sigma(\epsilon_0, \epsilon_1, \dots, \epsilon_L) = 0.$$

The stability analysis for linear prediction processes can be carried out using tools coming from wavelet theory, subdivision schemes and functional analysis (see [11]), however none of these techniques is applicable in general when the prediction process is nonlinear.

The discrete multiresolution framework of Harten [11] was developed to use nonlinear reconstruction processes. In signal and image examples [6], [4], [7], [9], we can see the nonlinear process allows a better adapted treatment of singularities. In these cases, stability can be

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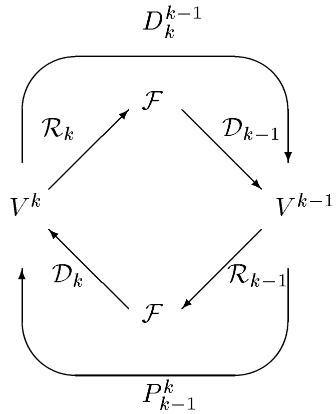

Figure 1: Definition of the operators

ensured by modifying the encoding algorithm. The idea of a modified-encoding to deal with nonlinear multiresolution schemes is due to Harten. One dimensional algorithms in several settings can be found in [10], [7]. The goal of a modified-encoding procedure is to keep track of the accumulation error in processing the values in the multi-scale representation. A synchronization of the encoding and decoding algorithms is obtained [5].

In this paper, we consider the hat average multiresolution setting [8]. In this framework, we can develop nonlinear schemes adapted to the presence of different types of discontinuities as δ's, jumps and corners [9]. The space of such functions is used, for instance, in vortex methods for the numerical solution of fluid dynamics problems.

In the framework of point values and cell averages we developed the stability for tensor product in [3]-[1] and in the non separable case in [2]. For a good adaptation to the singularities we have to consider the non separable approach.

The aim of this paper is to present non separable two-dimensional hat average multiresolution algorithms that ensure stability in the case of nonlinear prediction processes. We introduce a modified encoding for any reconstruction type. The multivariate context of tensor product emerges only as a particular case.

The paper is organized as follows: We recall the basic ingredients of the Harten's multiresolution framework in next section, focussing in the hat average setting 2.1. The error-control algorithms are discussed in 3. Finally, we give stability results in 4.

2. Harten's Framework

Harten's framework is based on two fundamental tools: discretization $\mathcal{D}_k$ and reconstruction $\mathcal{R}_k$. The discretization operator obtains discrete information from a (non-discrete) signal ($f \in \mathcal{F}$) at a particular resolution level k. The reconstruction operator, on the other hand, produces an approximation to a signal from its discrete values. This reconstruction can be nonlinear, and then better adapted to the considered problem.

Using these two operators we can connect linear vectors spaces (see figure 1), V^k, that represent in some way the different resolution levels (k increasing implies more resolution), i.e.,

$$\begin{aligned} D_k^{k-1} &: \quad V^k \to V^{k-1}, \quad \textit{decimation}, \\ P_{k-1}^k &: \quad V^{k-1} \to V^k, \quad \textit{prediction}. \end{aligned}$$

We focus on the specific case corresponding to the hat average discretization.