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MODIFIED LEGENDRE RATIONAL SPECTRAL METHOD FOR THE WHOLE LINE *1)

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Abstract

A mutually orthogonal system of rational functions on the whole line is introduced. Some approximation results are established. As an example of applications, a modified Legendre rational spectral scheme is given for the Dirac equation. Its numerical solution keeps the same conservation as the genuine solution. This feature not only leads to reasonable numerical simulation of nonlinear waves, but also simplifies the analysis. The convergence of the proposed scheme is proved. Numerical results demonstrate the efficiency of this new approach and coincide with the analysis well.

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Key words: Modified Legendre rational approximation, The whole line, Dirac equation.

1. Introduction

In sciences and engineerings, we often need to solve some problems in unbounded domain numerically, such as fluid flows in an infinite strip, nonlinear wave equations in quantum mechanics and so on. One of numerical methods for such problems is to use spectral approximations associated with certain orthogonal systems of polynomials in unbounded domains, such as the Hermite and the Laguerre approximations, see, Funaro and Kavian [8], Maday, Pernaud-Thomas and Vandeven [23], Guo [9], Guo and Shen [17] and Shen [25]. The next is to reform the original problems in unbounded domains and then use the Jacobi approximation to resolve the resulting singular problems in bounded domains numerically, see, Guo [10-13]. Another effective method is based on rational approximations. Boyd [5,6] and Christov [7] provided some spectral schemes for linear problems on infinite intervals by using certain mutually orthogonal systems of rational functions. Recently, Guo, Shen and Wang [18,19], Guo and Wang [21], and Wang and Guo [27] developed various rational approximations on infinite intervals. The rational spectral methods have several advantages. For instance, their weights are much weaker than the Hermite and Laguerre spectral methods and so it is not needed to reform the original problems usually. Moreover they are easier to be used for exterior problems than the Jacobi spectral methods. However, the non-uniform weights in the standard rational approximations may bring in some difficulties in actual computation in some applications. In particular, for the numerical simulations of hyperbolic systems, non-parabolic dissipative systems and nonlinear waves, such as the Schödinger equation, the Korteweg-de Vries equation and the Dirac equation etc.. Indeed the solutions of these systems satisfy some conservations which play important roles in theoretical analysis and numerical simulation. But the appearance of the non-uniform weights may destroy the corresponding conservations for the numerical solutions. This fact decreases the exactness of numerical experiments, and makes the numerical analysis

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complicated. To remedy this deficiency, Guo and Shen [17] proposed a modified Legendre rational approximation on the half line with the weight $\chi(x) \equiv 1$. The purpose of this paper is to develop a modified Legendre rational approximation on the whole line and its applications to numerical solutions of nonlinear wave equations. In this case, the numerical solutions keep the same conservations as in continuous cases. Meanwhile, the corresponding numerical analysis is simplified essentially.

This paper is organized as follows. In the next section, we introduce a mutually orthogonal system of rational functions on the whole line with the weight $\chi(x) \equiv 1$, and discuss its basic properties. We also recall some basic results on the Jacobi approximation, which will be used in the sequel. Then we study the modified Legendre rational approximation in Section 3, and the corresponding interpolation approximation in Section 4. Some approximation results are established, which form the mathematical foundation of the modified Legendre rational spectral method on the whole line. Section 5 is for some applications of this new approach. We take the Dirac equation on the whole line as an example to show how to use this method for nonlinear wave equations. The convergence of the proposed scheme is proved. Some numerical results are presented in the final section, which demonstrate the efficiency of this new approach, and coincide with the analysis well. It is easy to generalize the results of this paper to other nonlinear problems in multiple-dimensions.

2. Modified Legendre Rational Functions and Some Basic Results on Jacobi Approximation

2.1. Modified Legendre Rational Functions

Let $\Lambda = \{x \mid -\infty < x < \infty\}$ and $\chi(x)$ be certain weight function in the usual sense. Denote by $(u,v)_\chi$ and $||v||_\chi$ the inner product and the norm of the weighted space $L^2_\chi(\Lambda)$ respectively, i.e.,

$$(u,v)_\chi = \int_\Lambda u(x)v(x)\chi(x)dx, \quad ||v||_\chi = (v,v)^{\frac{1}{2}}_\chi.$$

Further let $\partial_x v(x) = \frac{\partial}{\partial x} v(x)$, etc.. For any non-negative integer m,

$$H^m_\chi(\Lambda) = \{v \mid \partial^k_x v \in L^2_\chi(\Lambda), 0 \le k \le m\}.$$

The inner product, the semi-norm and the norm of $H^m_\chi(\Lambda)$ are given by

$$(u,v)_{m,\chi} = \sum_{k=0}^{m} (\partial^k_x u, \partial^k_x v)_\chi,$$
$$|v|_{m,\chi} = ||\partial^m_x v||_\chi, \quad ||v||_{m,\chi} = (v,v)^{\frac{1}{2}}_{m,\chi},$$

respectively. For any real $r > 0$, we define the space $H^r_\chi(\Lambda)$ with the norm $||v||_{r,\chi}$ by space interpolation. If $\chi(x) \equiv 1$, then we denote $H^r_\chi(\Lambda)$, $|v|_{r,\chi}$, $||v||_{r,\chi}$, $||v||_\chi$ and $(u,v)_\chi$ by $H^r(\Lambda)$, $|v|_r$, $||v||_r$, $||v||$ and (u,v), respectively. In addition, $||v||_\infty = ||v||_{L^\infty(\Lambda)}$.

Let $L_l(y)$ be the Legendre polynomial of degree l, $l = 0,1,2\cdots$. They are the eigenfunctions of the singular Sturm-Liouville problem

$$\partial_y((1-y^2)\partial_y L_l(y)) + l(l+1)L_l(y) = 0, \ l = 0,1,2\cdots, \tag{2.1}$$

and satisfy the following recurrence relations

$$L_{l+1}(y) = \frac{2l+1}{l+1} y L_l(y) - \frac{l}{l+1} L_{l-1}(y), \ l \ge 1, \tag{2.2}$$
$$(2l+1)L_l(y) = \partial_y L_{l+1}(y) - \partial_y L_{l-1}(y), \ l \ge 1. \tag{2.3}$$

Besides

$$L_l(1) = 1, \quad L_l(-1) = (-1)^l, \quad \partial_y L_l(1) = \frac{1}{2} l(l+1), \quad \partial_y L_l(-1) = (-1)^{l+1} \frac{1}{2} l(l+1).$$