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NATURAL BOUNDARY INTEGRAL METHOD AND ITS NEW DEVELOPMENT *1)

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

In this paper, the natural boundary integral method, and some related methods, including coupling method of the natural boundary elements and finite elements, which is also called DtN method or the method with exact artificial boundary conditions, domain decomposition methods based on the natural boundary reduction, and the adaptive boundary element method with hyper-singular a posteriori error estimates, are discussed.

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Key words: Natural boundary integral, Artificial boundary, Domain decomposition, Hyper-singular a posteriori estimates.

1. Introduction

In many fields of scientific and engineering computing it is necessary to solve boundary value problems of partial differential equations over unbounded domains. The standard techniques such as the finite element method will meet some difficulties, even if they are very effective for bounded domains[1].

In recent twenty years many computational methods for solving problems over unbounded domains have been developed, such as: the infinite element method[19], the adaptive finite element method[1], the finite element method with approximate condition on an artificial boundary[7,9,22], the boundary element method[10], the coupling method of finite and boundary elements[21], the overlapping and non-overlapping domain decomposition methods, especially, the coupling and domain decomposition methods based on the natural boundary reduction[26−31], and so on. Each method has its advantages and disadvantages.

The natural boundary integral method and its coupling with the finite element method are suggested and developed by K. Feng, D. Yu and H. Han in early 1980 (see [4-6,9,20-22]). And then a very similar method, so-called DtN method or exact artificial boundary condition method, has also been devised by J.B. Keller and D. Givoli in later 1980 [12]. These methods are very important for solving many problems over unbounded domains. Up to now there have already been a lot of papers in this direction[11,14,15].

In this paper some new development of the natural boundary integral method is also presented. The method is applied to 3D problems, parabolic and hyperbolic equations, and anisotropic elliptic problems. Based on the natural boundary integral operators the overlapping and non-overlapping domain decomposition methods are developed for problems over

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unbounded domain, which have very wide application background[3,18]. Besides, using hyper-singular residuals as a posteriori error estimates, the adaptive boundary element method is developed[13,32].

The natural boundary integral equations and their related computational methods can also be applied to some semi-linear and nonlinear problems. We will discuss it in our forthcoming papers.

2. Natural Boundary Reduction and DtN Operators

Finite element methods are effective for bounded domains. For unbounded domains we need use boundary integral or boundary element methods. There are different ways for reducing problems to boundary integral equations, based on which some boundary integral methods are developed. For the same physical problem, there are different mathematical formulations, which are equivalent in the theory, but often have different effects in the computing practice.

Based on natural boundary reduction, natural boundary integral method is developed by Feng and Yu. The natural boundary integral method has distinctive advantages. This method preserves basic properties of the original problem, has the same variational principle as finite element method, and can be coupled with finite elements directly and naturally . Furthermore, natural integral equation on artificial boundary is an exact artificial boundary condition. Natural integral operator is just the Dirichlet to Neumann (DtN) operator. It plays a key role in domain decomposition methods, where it has an another name: the Steklov–Poincaré operator.

There is a relation between Dirichlet data u_0 and Neumann data u_n, it is the DtN map, or natural integral equation:

$$u_n = \mathcal{K} u_0, \qquad \text{on } \partial\Omega, \tag{1}$$

where $\mathcal{K}$ is DtN operator, i.e. the natural integral operator. It is a hyper-singular integral operator, a pseudo-differential operator with positive order. $\partial\Omega$ is the boundary of domain Ω.

The solution u is given by Poisson integral formula:

$$u = P u_0, \qquad \text{in } \Omega. \tag{2}$$

For some typical equations, when Ω is a half-plane or half-space, an interior or exterior circular or spherical domain, $\mathcal{K}$ and P can be obtained explicitly.

3. Artificial Boundary Conditions on Circle and Ellipse

Circular artificial boundary is a good selection for most 2-d exterior problems [20,22]. With natural boundary reduction, we get some DtN operators on the circle $\Gamma = \{(r,\theta) | r = R, 0 < \theta \le 2\pi\}$ as follows , where (r,θ) are polar coordinates and R is a constant (see Yu's books [26,29]).

For harmonic equation:

$$\mathcal{K} = -\frac{1}{4\pi R \sin^2\frac{\theta}{2}} *, \tag{3}$$

which satisfies $\mathcal{K}^2 = -\dfrac{\partial^2}{\partial s^2}$, $s = R\theta$.

Let

$$K(\theta) = -\frac{1}{4\pi \sin^2\frac{\theta}{2}}.$$

For biharmonic equation:

$$\mathcal{K} = \begin{bmatrix} \frac{1+\nu}{R^3}\delta''(\theta) - \frac{2}{R^3}K''(\theta) & \frac{1+\nu}{R^2}\delta''(\theta) + \frac{2}{R^2}K(\theta) \\ \frac{1+\nu}{R^2}\delta''(\theta) + \frac{2}{R^2}K(\theta) & -\frac{1+\nu}{R}\delta(\theta) + \frac{2}{R}K(\theta) \end{bmatrix} *, \tag{4}$$

where ν is Poisson ratio.