Journal of Computational Mathematics, Vol.22, No.2, 2004, 275–286.

ON APPROXIMATION OF LAPLACIAN EIGENPROBLEM OVER A REGULAR HEXAGON WITH ZERO BOUNDARY CONDITIONS *1)

Jia-chang Sun
(*Parallel Computing Division, Institute of Software, Chinese Academy of Sciences
Beijing 100080, China*)
(*E-mail: sunjc@mail.rdcps.ac.cn*)

Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

In my earlier paper [4], an eigen-decompositions of the Laplacian operator is given on a unit regular hexagon with periodic boundary conditions. Since an exact decomposition with Dirichlet boundary conditions has not been explored in terms of any elementary form. In this paper, we investigate an approximate eigen-decomposition. The function space, corresponding all eigenfunction, have been decomposed into four orthogonal subspaces. Estimations of the first eight smallest eigenvalues and related orthogonal functions are given. In particulary we obtain an approximate value of the smallest eigenvalue $\lambda_1 \backsim \frac{29}{40}\pi^2 = 7.1555$, the absolute error is less than 0.0001.

Mathematics subject classification: 42B35, 65N25, 65T50
Key words: Laplacian eigen-decomposition, Regular hexagon domain, Dirichlet boundary conditions.

1. Introduction

Given an origin point O and two plane vectors e_1 and e_2, e.g. $e_1 = \{0, -1\}$, $e_2 = \{\frac{\sqrt{3}}{2}, \frac{1}{2}\}$, we form a 3-direction 2-D partition as drawn in Fig.1. To deal with symmetry along the three direction, we apply a 3-direction coordinates instead of the usual 2-D Cartesian coordinates and barycentry coordinates, which is a very useful within a triangle domain. Setting the origin point $O = (0, 0, 0)$, each partition line is represented by t_l=integer (l=1,2,3), and each 2-D point P is represented by

$$P = (t_1, t_2, t_3), \quad t_1 + t_2 + t_3 = 0, \tag{1.1}$$

and any function $f(P)$ defined on the plane can be written as $f(P) = f(t_1, t_2, t_3)$. In particulary, P_k is called an integer node if and only if $P_k = (k_1, k_2, k_3), \quad k_1 + k_2 + k_3 = 0$.

Let Ω be the unit regular hexagon domain

$$\Omega = \{P | P = (t_1, t_2, t_3) \qquad t_1 + t_2 + t_3 = 0, \quad -1 \le t_1, t_2, t_3 \le 1\} \tag{1.2}$$

we consider the following eigenvalue problem

$$-\Delta u = \lambda u, \tag{1.3}$$

with zero Dirichlet boundary

$$u|_{\partial\Omega} = 0 \tag{1.4}$$

In terms of the 3-direction partition form, the Laplacian operator can be written as

$$\mathcal{L} = -\frac{2}{3}\Delta = -(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2})^2 - (\frac{\partial}{\partial t_2} - \frac{\partial}{\partial t_3})^2 - (\frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_1})^2. \tag{1.5}$$

* Received January 31, 2004.
1) Project supported by the Major Basic Project of China (No.G19990328) and National Natural Science Foundation of China (No. 60173021).

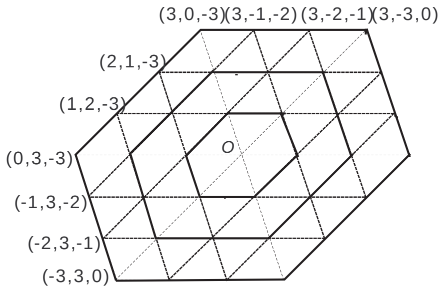

Figure 1.1: 3-direction partition

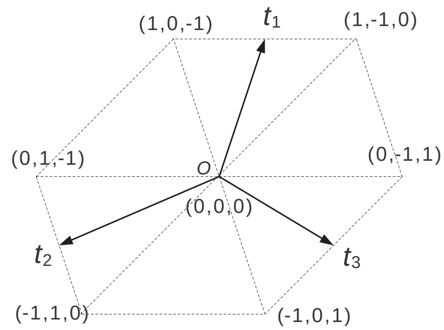

Figure 1.2: Parallel hexagon domain Ω

Definition 1.1. *A function $f(P)$, defined in the 3-direction coordinate, is called periodic, if for all $P = (t_1, t_2, t_3)$ the equality*

$$f(P+Q) = f(P)$$

holds for any integer vector $Q = (n_1, n_2, n_3)$ with $n_1^2 + n_2^2 + n_3^2 = 0 \pmod 6$

The following results have been proved in my earlier paper [4].

Theorem 1.1. *For all integer triple $j = (j_1, j_2, j_3)$, the function system*

$$g_j(P) = e^{i\frac{2\pi}{3}(j_1t_1+j_2t_2+j_3t_3)}$$

forms a complex eigen decomposition of the Laplacian operator (1.3) with three direction periodic boundary conditions, where $P = (t_1, t_2, t_3)$, $j_1 + j_2 + j_3 = 0$. The corresponding eigenvalues equal to

$$\lambda_{j_1,j_2,j_3} = (\frac{2\pi}{3})^2((j_1-j_2)^2 + (j_2-j_3)^2 + (j_3-j_1)^2) \tag{1.6}$$

Since

$$\lambda_{j_1,j_2,j_3} = \lambda_{j_2,j_3,j_1} = \lambda_{j_3,j_1,j_1} = \lambda_{-j_1,-j_3,-j_2} = \lambda_{-j_3,-j_2,-j_1} = \lambda_{-j_2,-j_1,-j_3}$$

we have the following real eigen-decomposition.

Corollary 1.1. *For all integer triple $j = (j_1, j_2, j_3)$, two function system*

$$\cos(\frac{2\pi}{3}(j_1t_1 + j_2t_2 + j_3t_3)) \quad \textit{and} \quad \sin(\frac{2\pi}{3}(j_1t_1 + j_2t_2 + j_3t_3))$$

form an eigen decomposition of the Laplacian operator (1.3) with three direction periodic boundary conditions. Moreover, except the smallest eigenvalue is single, all other eigenvalues are six multiple.

It is clear that the first eigenfunction is a trivial constant. Several figures of eigenfunctions, related from the second to the four without counting multiple, are drawn in Figure 1.3- 1.12.

Definition 1.2. $TSin_j(P) := \frac{1}{2i}\big[g_{j_1,j_2,j_3}(P) + g_{j_2,j_3,j_1}(P) + g_{j_3,j_1,j_2}(P)$

$$-g_{-j_1,-j_3,-j_2}(P) - g_{-j_2,-j_1,-j_3}(P) - g_{-j_3,-j_2,-j_1}(P)\big] \tag{1.7}$$

Definition 1.3. $TCos_j(P) := \frac{1}{2}\big[g_{j_1,j_2,j_3}(P) + g_{j_2,j_3,j_1}(P) + g_{j_3,j_1,j_2}(P)$

$$+g_{-j_1,-j_3,-j_2}(P) + g_{-j_2,-j_1,-j_3}(P) + g_{-j_3,-j_2,-j_1}(P)\big] \tag{1.8}$$