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THREE DIMENSION QUASI-WILSON ELEMENT FOR FLAT HEXAHEDRON MESHES *1)

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Dedicated to Professor Zhong-ci Shi on the occasion of his 70th birthday

Abstract

The well known Wilson's brick is only convergent for regular cuboid meshes. In this paper a quasi-Wilson element of three dimension is presented which is convergent for any hexahedron meshes. Meanwhile the element is anisotropic, that is it can be used to any flat hexahedron meshes for which the regular condition is unnecessary.

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1. Introduction

The classical finite element approximation relies on the regular [5] or nondegenerate [4] condition,ie. there exists a constant c such that

$$h_k/e_k \le c, \quad \forall K \tag{1.1}$$

where h_K is diameter of K and ρ_K is diameter of the biggest ball contained in K. But recently some researches [2,3,7,18] show that the condition (1.1) is not necessary for the convergence of some finite elements, i.e, these elements can be well used in narrow meshes.

The well known Wilson's elements are nonconforming elements for the problems of order two. However the two dimension Wilson's element is only convergent for rectangular and parallelogram meshes. In order to extend this element to arbitrary quadrilateral meshes, various improved methods have been developed, see [6,7,10,11,12,13,14,15,17,18]. But seldom papers consider the three dimension Wilson's element. In the same way the three dimension Wilson's element is only convergent for regular cuboid meshes. In this paper a quasi-Wilson element of three dimension is presented. We prove that this element is convergent for any flat hexahedron meshes, this means its convergence is independent of regular(1.1).

2. Three Dimension Quasi-Wilson Element

Let $\hat{K} = [-1,1]^3$ be the reference element with vertices $\hat{A}_i(\hat{a}_{1i}, \hat{a}_{2i}, \hat{a}_{3i}), 1 \le i \le 8$, where $(\hat{a}_{11} \cdots \hat{a}_{18}) = (-1,1,1,-1,-1,1,1,-1)$ $(\hat{a}_{21}, \cdots, \hat{a}_{28}) = (-1,-1,1,1,-1,-1,1,1)$, $(\hat{a}_{31}, \cdots, \hat{a}_{38}) = (-1,-1,-1,-1,1,1,1,1)$. We define on $\hat{K}$ the finite element $(\hat{K}, \hat{p}, \hat{\sum})$ as following:

$$\hat{P} = span\{\hat{N}_1, \cdots, \hat{N}_8, \hat{\phi}(\hat{x}_1), \hat{\phi}(\hat{x}_2), \hat{\phi}(\hat{x}_3)\} \tag{2.1}$$

where

$$\hat{N}_i = \frac{1}{8}(1+\hat{a}_{1i}\hat{x}_1)(1+\hat{a}_{2i}\hat{x}_2)(1+\hat{a}_{3i}\hat{x}_3), 1 \le i \le 8, \hat{\phi}(t) = -\frac{3}{32}(t^2-1) + \frac{5}{64}(t^4-1)$$

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When $\phi(t) = \frac{1}{16}(t^2 - 1)$, it is Wilson's brick. Obviously $Q_1(\hat{k}) = span\{\hat{N}_1, \cdots, \hat{N}_8\}$, and $\hat{N}_i(\hat{A}_j) = \delta_{ij}, 1 \le i, j \le 8$

$$\hat{\sum} = \{\hat{v_1}, \cdots, \hat{v_8}, g_1(\hat{v}), g_2(\hat{v}), g_3(\hat{v})\} \tag{2.2}$$

where $\hat{v_i} = \hat{v}(\hat{A}_i), \quad 1 \le i \le 8, \quad g_j(\hat{v}) = \int_{\hat{K}} \frac{\partial^2 \hat{v}}{\partial \hat{x}_j^2} d\hat{x}, \quad 1 \le j \le 3$. It is easy to see that

$$\forall \quad \hat{v} \in \hat{P}, \quad \hat{v} = \hat{v}^0 + \hat{v}^1 \tag{2.3}$$

where

$$\hat{v}^0 = \sum_{i=1}^{8} \hat{v_i}\hat{N}_i(\hat{x}), \quad \hat{v}^1 = \sum_{i=1}^{3} \hat{g}_i(\hat{v})\hat{\phi}(\hat{x_i}), \tag{2.4}$$

Let K be a convex hexahedron with vertices $A_i(a_{1i}, a_{2i}, a_{3i}), 1 \le i \le 8$. The mapping

$$F_K \in (\hat{Q_1}^3), F_K(\hat{x}) = (x_1^K(\hat{x}), x_2^K(\hat{x}), x_3^K(\hat{x})), x_i^K(\hat{x}) = \sum_{j=1}^{8} \hat{N}_j(\hat{x})a_{ij}, \quad 1 \le i \le 8 \tag{2.5}$$

makes $F_K(\hat{K}) = K, F_K(\hat{A}_i) = A_i, 1 \le i \le 8$

For any function $v(x)$ defined on K, we define $\hat{v}(\hat{x})$ by

$$\hat{v}(\hat{x}) = \hat{v}(x^K(\hat{x})), \quad or \quad \hat{v} = v \circ F_K$$

On the hexahedron element K, we define the shape function space P_K,

$$P_K = \{p = \hat{p} \circ F_K^{-1}; \hat{p} \in \hat{P}\}$$

Given a convex polyhedron domain Ω, let $\bar{\Omega} = \bigcup_{K \in T_h} K$ be a decomposition T_h of $\bar{\Omega}$. The finite-element space is defined by $V_h = \{v; v|_K \in P_K, \forall K \in T_h$; v is continuous at the vertices of elements and vanishing at the vertices on the boundary of $\Omega\}$.

Consider the model problem,

$$\begin{cases} -\Delta u = f & in \ \ \Omega \\ u = 0 & on \ \ \partial\Omega \end{cases} \tag{2.6}$$

where Ω is a bounded, convex polyhedron in three dimension.

Its weak form is find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega) \tag{2.7}$$

where $a(u, v) = \int_\Omega \nabla u . \nabla v dx, \quad f(v) = \int_\Omega f v dx$. By theorem 1.8 of [8] (see [9]), when $f \in L^p(\Omega), p > 2$,

$$u \in W^{2,p}(\Omega) \tag{2.8}$$

The quasi-Wilson element approximation of (2.7) is defined by find $u_h \in v_h$ such that

$$a_h(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h \tag{2.9}$$

where $a_h(u_h, v_h) = \sum_k \int_k \nabla u_h . \nabla v_n dx$. Since$V_h$ is not contained in $H_0^1(\Omega), V_h$ is a nonconforming approximation of $H_0^1(\Omega)$

For every $v_h \in V_h$, we define

$$|v_n|_{1,h}^2 = a_h(v_h, v_h)$$

It is easy to check that $|\cdot|_{1,h}$ is a norm over V_h. Every $v_h \in V_h$ can be written as

$$v_h = v_h^0 + v_h^1 \tag{2.10}$$

where $\forall K \in T_h, F_K : \hat{K} \to K$, with

$$v_h^0|_K = \sum_{i=1}^{8} \hat{N}_i(\hat{x})v_h(A_i) = \hat{v}^0 \circ F_K^{-1}, \quad v_h^1|_K = \sum_{i=1}^{3} \hat{\phi}_i(\hat{x_i})g_i(\hat{v}) = \hat{v}^1 \circ F_K^{-1}$$