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THE LONG-TIME BEHAVIOR OF SPECTRAL APPROXIMATE FOR KLEIN-GORDON-SCHRÖDINGER EQUATIONS *1)

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Abstract

Klein-Gordon-Schrödinger (KGS) equations are very important in physics. Some papers studied their well-posedness and numerical solution [1-4], and another works investigated the existence of global attractor in R^n and $\Omega \subset R^n$ $(n \leq 3)$ [5-6,11-12]. In this paper, we discuss the dynamical behavior when we apply spectral method to find numerical approximation for periodic initial value problem of KGS equations. It includes the existence of approximate attractor $\mathcal{A}_N$, the upper semi-continuity on $\mathcal{A}$ which is a global attractor of initial problem and the upper bounds of Hausdorff and fractal dimensions for $\mathcal{A}$ and $\mathcal{A}_N$, etc.

Key words: Klein-Gordon-Schrödinger equation, Spectral approximate, Global attractor, Hausdorff dimension, Fractal dimension.

1. Introduction

In this paper, we consider the following periodic initial value problem of dissipative KGS equations

$$\begin{cases} i\psi_t + \triangle\psi + i\nu\psi + \phi\psi = f(x), & x \in I, I = [0, 2\pi]^3, \\ \phi_{tt} + \gamma\phi_t - \triangle\phi + \phi - |\psi|^2 = g(x), & t > 0, \end{cases}$$

where ψ, ϕ are complex and real unknown functions, respectively, ν, γ are positive constants, f and g are given complex and real functions respectively, $i\nu\psi$ and $\gamma\phi_t$ are dissipative terms. We introduce the transformation $\theta = \phi_t + \delta\phi$, where δ is a small positive constant and then the above problem can be written as

$$\begin{cases} i\psi_t + \triangle\psi + i\nu\psi + \phi\psi = f(x), & (1.1) \\ \phi_t + \delta\phi = \theta, \qquad x \in I, t > 0, & (1.2) \\ \theta_t + (\gamma - \delta)\theta - \triangle\phi + (1 - \delta(\gamma - \delta))\phi - |\psi|^2 = g(x), & (1.3) \\ \psi(x + 2\pi e_i, t) = \psi(x, t), \phi(x + 2\pi e_i, t) = \phi(x, t), \theta(x + 2\pi e_i, t) = \theta(x, t), & (1.4) \\ \psi(x, 0) = \psi_0(x), \phi(x, 0) = \phi_0(x), \theta(x, 0) = \theta_0(x), & (1.5) \end{cases}$$

where e_i are unit vectors in the i-th direction.

Let $H_p^s(I)$ denote the periodic real or complex Sobolev space with the inner product $(\cdot,\cdot)_s$ and the norm $\|\cdot\|_s$. In particular, $H_p^0(I) = H(I)$, and its inner product and norm are $(\cdot,\cdot)$, $\|\cdot\|$, respectively. Denote $V = H_p^1(I) \times H_p^1(I) \times H(I)$, $\|(\psi,\phi,\theta)\|_V^2 = \|\psi\|_1^2 + \|\phi\|_1^2 + \|\theta\|^2$, $\widetilde{V} = H_p^2(I) \times H_p^2(I) \times H_p^1(I)$, $\|(\psi,\phi,\theta)\|_{\widetilde{V}}^2 = \|\psi\|_2^2 + \|\phi\|_2^2 + \|\theta\|_1^2$.

Assume that $S_N = \text{span}\{e^{ix\cdot j} | j \in Z^3, |j| \leq N\}$, P_N is an orthogonal projection from L^2 to S_N.

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Lemma 1[7]**.** *For any $\sigma\geq 0$, if $u\in H_p^\sigma(I)$ then*

$$\|u-P_N u\|_j\leq cN^{-\sigma+j}\|u\|_\sigma,\qquad 0\leq j\leq\sigma.$$

Lemma 2. *For $u\in H_p^1(I)$, $n=3$, then*

$$\|u\|_{L^4}\leq c\|u\|_1^{3/4}\|u\|^{1/4}.$$

2. Some Results on the Problem (1.1)-(1.5)

Lemma 3. *Assume that f , $g\in L^2(I)$ and $\|(\psi_0,\phi_0,\theta_0)\|_V\leq R$, then there exists a constant $\delta_1>0$, such that if $\delta\leq\delta_1$ then solution of the problem (1.1)-(1.5) satisfies*

$$\|\psi(t)\|_1+\|\phi(t)\|_1+\|\theta(t)\|\leq D_1, t\geq t_1,$$

where D_1 depends on $\nu,\gamma,\delta,\|f\|,\|g\|$; t_1 depends on $\nu,\gamma,\delta,\|f\|,\|g\|$ and R.
Lemma 4. *Assume that f , $g\in H_p^k(I)$, $k\geq 0$ and $\|\psi_0\|_{k+2}+\|\phi_0\|_{k+2}+\|\theta_0\|_{k+1}\leq R$, then the solution $(\psi,\ \phi,\ \theta)$ of the problem satisfies*

$$\|\psi(t)\|_{k+2}+\|\phi(t)\|_{k+2}+\|\theta(t)\|_{k+1}\leq D_{k+2},\qquad t\geq t_{k+2},$$

where D_{k+2} depends on $\nu,\gamma,\delta,\|f\|_k,\|g\|_k$ and k ; t_{k+2} depends on $\nu,\gamma,\delta,\|f\|_k,\|g\|_k$, k and R.

The proof of Lemma 3 and Lemma 4 are similar the paper[5] we don't represent.

Furthermore, we can prove

Lemma 5. *Under the conditions of Lemma 3, for $0\leq t\leq T$, we have*

$$\|\psi(t)\|_1+\|\phi(t)\|_1+\|\theta(t)\|\leq L_1,$$

where L_1 depends on $\nu,\gamma,\delta,\|f\|,\|g\|$, and T.
Lemma 6. *Under the conditions of Lemma 4, for $0\leq t\leq T$, we have*

$$\|\psi(t)\|_{k+2}+\|\phi(t)\|_{k+2}+\|\theta(t)\|_{k+1}\leq L_{k+2},$$

where L_{k+2} depends on $\nu,\gamma,\delta,\|f\|_k,\|g\|_k$, and T.

By the above Lemmas and the theory of partial differential equation we can obtain that the problem (1.1)-(1.5) defineds a continuous operator semigroup $\{S(t)\}_{t\geq 0}$, $S(t)(\psi_0,\phi_0,\theta_0)=(\psi(t),\phi(t),\theta(t))$. If we denote

$$B_1=\{(\psi,\phi,\theta)\in V|\ \|\psi\|_1+\|\phi\|_1+\|\theta\|\leq M_1\}$$

and

$$B_k=\{(\psi,\phi,\theta)\in H_p^{k+2}\times H_p^{k+2}\times H_p^{k+1}|\ \|\psi\|_{k+2}+\|\phi\|_{k+2}+\|\theta\|_{k+1}\leq M_{k+2}\},$$

where $M_i,\ i=1,2,...,k+2$ are proper large constants, then B_1 and B_k are bounded absorbing sets on V and $H_p^{k+2}\times H_p^{k+2}\times H_p^{k+1}$ respectively. We can also prove the following result using the technique introduced by Temam[8].

Theorem 1. *Suppose that f,$g\in H$, then the problem (1.1)-(1.5) has a global attractor $\mathcal{A}$ on V,which is a compact invariant subset of V,absorbs any bounded set of V.*

If f, $g\in H_p^k(I)$, $k\geq 0$, then (1.1)-(1.5) has a global attractor on $Q^k=H_p^{k+2}\times H_p^{k+2}\times H_p^{k+1}$ which is a compact invariant subset of Q^k and absorbs any bounded set of Q^k.

The proof of Theorem is similar to [5]. We can prove $S(t)$ is asymptotically compact in Q^k, that is if (ψ_n,ϕ_n,θ_n) is bounded in Q^k and $t_n\longrightarrow\infty$, then $S(t_n)(\psi_n,\phi_n,\theta_n)$ is precompact in Q^k, thus from the Theorem I.1.1 of [8], we obtain the existence of global attractor $\mathcal{A}_k$ in Q^k.