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THE DERIVATIVE PATCH INTERPOLATING RECOVERY TECHNIQUE FOR FINITE ELEMENT APPROXIMATIONS *1)

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Abstract

A derivative patch interpolating recovery technique is analyzed for the finite element approximation to the second order elliptic boundary value problems in two dimensional case. It is shown that the convergence rate of the recovered gradient admits superc onvergence on the recovered subdomain, and is two order higher than the optimal global convergence rate (ultracovergence) at an internal node point when even order finite element spaces and local uniform meshes are used.

Key words: Finite element, Derivative recovery, Ultraconvergence.

1. Introduction

Finite element superconvergence property has long attracted considerable attentions since its practical importance in enhancing the accuracy of finite element calculation and in adaptive computing via a posteriori error estimate. In this field affluent research results have been achieved. For some complete literature on superconvergence, the reader is referred to Wahlbin's book [1], Chen and Huang's book [2], and a recent conference proceeding edited by Krizek et al. [3]. In article [4,5], Lin Qun et al. proposed a new type of interpolation operator into the finite element spaces, that is the interpolation operator of projection type, and remarked that it will approximate the finite element solutions much better than the usual Lagrange interpolation. Thus, the interpolation operator of projection type provides a new powerful means in the research of finite element superconvergence, and we will use it as main analysis means in this paper.

In a previous work[6], the ultraconvergence (i.e., two order higher than the optimal global convergence rate) of the derivative patch interpolating recovery technique was analyzed for a class of two-point boundary value problems. The current work is devoted to the superconvergence and ultraconvergence properties of the derivative interpolating recovery technique for finite element approximation to the elliptic equation $Au = f$ on a rectangular domain with the general partial differential operator of second order

$$A = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} a_{ij} \frac{\partial}{\partial x_i} + \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} + a_0 I$$

and $A = -\Delta + a_0 I$ when ultraconvergence is concerned. In this article, we will assume that the rectangular partition mesh is regular for general case, or quasi-regular when superconvergence is considered. Moreover, when we analysis the ultraconvergence at an interior nodal point p_0, we will also assume that the mesh is local uniform, that is the four elements which share

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the common interior nodal point p_0 are uniform. In general, the solution of the second order elliptic boundary value problems on a rectangular domain may have corner singularity, and consequently, the finite element approximation may suffer from the "pollution effect" which will result in the failure of the recovery procedure. There have been many techniques to treat the pollution effect caused by domain singularity, for example, the local mesh refinement. In order to concentrate on the local recovery method, in this paper, we assume that the solution is smooth enough on domain Ω for our purpose, otherwise some local analysis methods should be used[4,7].

Recently, many research works focus on the so-called $Z-Z$ derivative patch recovery technique[8-11], and this technique is considered to be one of the most effectiveness techniques in the research of asymptotically exact a posteriori error estimates[12]. This technique uses the least square method to fit the first order derivatives of finite element solution and results in superconvergence. The ultraconvergence property of $Z-Z$ technique has been analyzed by Zhang[13] for the Laplace equation in the two dimensional setting. Comparing with the $Z-Z$ technique, our recovery method is more simple and easier to implement, and possesses the explicit expression.

In this paper, we shall use notation H_0^1 and W_p^m to represent the usual Sobolev spaces on domain Ω with norm and seminorm $\|\cdot\|_{m.p}$ and $|\cdot|_{m.p}$ on W_p^m, respectively, and use letter C to denote a generic constant.

The plan of this paper is as follows: In Section 2 we introduce the interpolation operator of projection type and discuss its approximation properties. In Section 3 the derivative patch interpolating recovery operator is defined and its super-approximation and ultra-approximation properties are analyzed. Section 4 is devoted to the superconvergence and ultraconvergence properties for the finite element approximation to the second order elliptic boundary value problems.

2. Interpolation Operator of Projection Type and Its Super-approximation Properties

Let element $e=e_1\times e_2=(x_e-h_e,x_e+h_e)\times(y_e-\hbar_e,y_e+\hbar_e)$, $\{L_j(x)\}_{j=0}^{\infty}$ and $\{\widetilde{L}_j(y)\}_{j=0}^{\infty}$ be the normalized orthogonal *Legendre* polynomial systems on $L_2(e_1)$ and $L_2(e_2)$, respectively. Set

$$\omega_0(x)=\widetilde{\omega}_0(y)=1,\quad \omega_{j+1}(x)=\int_{x_e-h_e}^{x}L_j(x)\,dx,\ \ \widetilde{\omega}_{j+1}(y)=\int_{y_e-\hbar_e}^{y}L_j(y)\,dy,\ \ j\geq 0$$

It is well known that polynomials $\omega_{k+1}(x)$ and $L_k(x)\,(k\geq 1)$ have $k+1$ and k zero points on $\overline{e}_1$ and e_1, respectively, and these zero points are symmetrically distributed with respect to the middle point x_e. Denote the two kinds of zero point set by $N_k^{(0)}=\{\,g_j^{(0)}\,\}$ and $N_k=\{\,g_j\,\}$, respectively, and we call $N_k^{(0)}$ the Lobatto point set and N_k the Gauss point set. Moreover, we know that these polynomials also possess the following symmetry and antisymmetry

$$\omega_{2j}(x_e+x)=\omega_{2j}(x_e-x),\quad \omega_{2j-1}(x_e+x)=-\omega_{2j-1}(x_e-x)\tag{1}$$

$$L_{2j}(x_e+x)=L_{2j}(x_e-x),\quad L_{2j-1}(x_e+x)=-L_{2j-1}(x_e-x)\tag{2}$$

The completely parallel conclusions hold for the polynomials $\widetilde{\omega}_{k+1}(y)$ and $\widetilde{L}_k(y)$ on element $e_2=(y_e-\hbar_e,y_e+\hbar_e)$.

Below we denote the Lobatto and Gauss points on element $e=e_1\times e_2$ by $\{\,G_{ij}^{(0)}=(g_i^{(0)},\widetilde{g}_j^{(0)})\,\}$ and $\{\,G_{ij}=(g_i,\widetilde{g}_j)\,\}$, respectively, and also denote the *Gauss* lines by $G_{x,j}=\{\,(x,\widetilde{g}_j);\ x\in e_1,\ \widetilde{g}_j\in\widetilde{N}_k\}$ and $G_{i,y}=\{\,(g_i,y)\,;\,g_i\in N_k,\ y\in e_2\,\}$.