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SOME NONLINEAR APPROXIMATIONS FOR MATRIX-VALUED FUNCTIONS *1)

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Abstract

Some nonlinear approximants, i.e., exponential-sum interpolation with equal distance or at origin, (0,1)-type, (0,2)-type and (1,2)-type fraction-sum approximations, for matrix-valued functions are introduced. All these approximation problems lead to a same form system of nonlinear equations. Solving methods for the nonlinear system are discussed. Conclusions on uniqueness and convergence of the approximants for certain class of functions are given.

Key words: Matrix-valued function, Nonlinear approximation, Interpolation.

1. Introduction

For scalar functions, Baker-Gammel approximation was developed in a series of papers [1, 2, 3, 7] (see [7] for more references) as a method for producing nonlinear approximation with good convergence properties. The main features of these methods can be described as follows: **1**. Use a generating function for the function to be approximated to derive a nonlinear approximant by agreement condition between the function and the approximant. **2**. Draw on the relations between the nonlinear approximants and Padé approximants. **3**. Establish convergence results from this relation and the convergence results of Padé approximation.

In the present paper, we make efforts to generalize these ideas to matrix-valued functions. A few concrete nonlinear approximations, such as exponential sum interpolation at origin and Padé-like approximation, are discussed. Since the multiplications of matrices are not commuting, this generalization is not straightforward. The first problem is how to do partial fraction for a matrix rational function. In the aspect of existence and uniqueness of the nonlinear approximants, there also exist some problems that unlike their scalar counterpart. We shall give solutions to these problems in this paper under certain conditions.

The remaining of the paper is organized as follows: We first introduce the definitions of the nonlinear approximation problems in §2. All these problems lead to a same form system of nonlinear equations. Then we solve this system in §3 using the matrix Padé approximation method and the theory of matrix polynomials. In §4, we establish the relationship between these approximants defined in §2, and then we discuss the uniqueness problem in §5. Finally, utilizing the convergence results of the Padé approximation for the matrix-valued Stieltjes function, we give the convergence conclusion of the nonlinear approximants. Some terminologies and basic facts in the matrix polynomial theory used in this paper are provided in Appendix.

2. Definition of the Approximants

Now we define our nonlinear approximants for matrix-valued functions.

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a. Exponential-sum interpolation with equal distance.

Let $f(z) \in \mathbb{C}^{n\times n}[z]$ be a given matrix-valued function. Construct a function in the form

$$F(z) = \sum_{j=1}^{l} A_j e^{s_j z},$$

where A_j, $s_j \in \mathbb{C}^{n\times n}$ are parameters to be determined for $j = 1, \cdots, l$, such that

$$F(iT) = f(iT), \quad i = 0, 1, \cdots, 2l-1, \tag{2.1}$$

where T is a given constant. Put $c_i = f(iT), S_j = e^{s_j T}$ for $j = 1, \cdots, l$, then (2.1) can be written as

$$\sum_{j=1}^{l} A_j S_j^i = c_i, \quad i = 0, 1, \cdots, 2l-1. \tag{2.2}$$

Therefore, if the solutions of equation(2.2) are found for the unknowns A_j and S_j, $j = 1, \cdots, l$, then s_j can be determined by

$$s_j = \frac{1}{T}\log S_j, \quad j = 1, 2, \cdots, l.$$

b. Exponential-sum interpolation at the origin.

For a given power series $f(z) = \sum_{i=0}^{\infty} \frac{c_i}{i!} z^i \in \mathbb{C}^{n\times n}[z]$, find

$$E(z) = \sum_{i=0}^{J} \frac{B_i}{i!} z^i + \sum_{j=1}^{l} A_j e^{S_j z} \tag{2.3}$$

such that

$$\left.\frac{d^i E(z)}{dz^i}\right|_{z=0} = c_i, \quad i = 0, 1, \cdots, 2l+J, \quad J \geq -1. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\begin{cases} B_i + \sum_{j=1}^{l} A_j S_j^i = c_i, & i = 0, 1, \cdots, J, \\ \sum_{j=1}^{l} A_j S_j^i = c_i, & i = J+1, \cdots, 2l+J. \end{cases} \tag{2.5}$$

If $J = -1$, the system of equations (2.5) is the same as (2.2).

c. (0,1)-type fraction-sum approximation.

Let $f(z) = \sum_{i=0}^{\infty} c_i z^i$ be a given power series with $c_i \in \mathbb{C}^{n\times n}$. The problem of (0,1)-type fraction-sum approximation is to find a function in the form

$$P(z) = \sum_{i=0}^{J} B_i z^i + \sum_{j=1}^{l} A_j (I - zS_j)^{-1}, \quad J \geq -1 \tag{2.6}$$

such that

$$f(z) - P(z) = O(z^{2l+J+1}), \quad z \to 0. \tag{2.7}$$