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A NEW SMOOTHING EQUATIONS APPROACH TO THE NONLINEAR COMPLEMENTARITY PROBLEMS [*1)]

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Abstract

The nonlinear complementarity problem can be reformulated as a nonsmooth equation. In this paper we propose a new smoothing Newton algorithm for the solution of the nonlinear complementarity problem by constructing a new smoothing approximation function. Global and local superlinear convergence results of the algorithm are obtained under suitable conditions. Numerical experiments confirm the good theoretical properties of the algorithm.

Key words: Nonlinear complementarity problem, Smoothing Newton method, Global convergence, Superlinear convergence.

1. Introduction

Let $F : R^n \to R^n$ be a continuously differentiable mapping and X be a nonempty closed convex set in R^n. The variational inequality problems, denoted by $\mathrm{VIP}(F, X)$, is to find a vector $x^* \in X$ such that

$$F(x^*)^T(x - x^*) \geq 0 \quad \text{for all } x \in X \tag{1.1}$$

If $X = R^n_+$, $\mathrm{VIP}(F, X)$ reduces to the nonlinear complementarity problem, denoted $\mathrm{NCP}(F)$, which is to find $x \in R^n$ such that

$$x \geq 0, F(x) \geq 0, x^T F(x) = 0. \tag{1.2}$$

Two comprehensive surveys of variational inequality problems and nonlinear complementarity problems are [1] and [3]. The study on iterative methods for solving $\mathrm{VIP}(F, X)$ and $\mathrm{NCP}(F)$ has been rapidly developed in the last decade. One of the most popular approaches is to reformulate $\mathrm{NCP}(F)$ as an equivalent nonsmooth equation so that generalized Newton-type methods can be applied in a way similar to those for smooth equations.

Much effort has been made to construct smoothing approximation functions for approach to the solution of $\mathrm{NCP}(F)$ in recent years [2, 4, 5, 6, 7, 18, 19]. This class of algorithms, called smoothing Newton method, is due to Chen, Qi, and Sun [2]. In [2], the locally superlinear

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convergence of a smoothing Newton method is established. In this paper, we will construct a new smoothing approximation function and present a new smoothing Newton method. The proposed smoothing Newton method meets the demands used in the Chen et al. in [2] and is easy to implement. We will show global and superlinear convergence of the proposed method under the same assumptions as used by Chen et al. [2] and by Qi et al. [19].

Next we introduce some words about our notation: Let $G : R^n \to R^m$ be continuously differentiable. The $\nabla G(x) \in R^{m\times n}$ denotes the Jacobian of G at a point $x \in R^n$. If $m = 1$, $\nabla G(x)$ denotes the gradient of G at a point $x \in R^n$. If is $G : R^n \to R^m$ only local Lipschitzian, we can define Clarke's [12] generalized Jacobian as follows:

$$\partial G(x) := \text{conv}\{H \in R^{m\times n} | \exists \{x^k\} \subseteq D_G : x^k \to x \text{ and } G'(x^k) \to H\};$$

here D_G denotes the set of differentiable points of G and conv$S$ is the convex hull of a set S. If $m = 1$, we call $\partial G(x)$ the generalized gradient of G at x for obvious reasons.

Usually, $\partial G(x)$ is not easy to compute, especially for $m > 1$. Based on this reason, we use in this paper a kind of generalized Jacobian for the function G, denoted by $\partial_C G$ and defined as(see [13])

$$\partial_C G = \partial G_1(x) \times \partial G_2(x) \times \cdots \times \partial G_n(x),$$

where $G_i(x)$ is ith component function of G.

Furthermore, we denote by $\|x\|$ the Euclidian norm of x if $x \in R^n$ and by $\|A\|$ the spectral norm of a matrix $A \in R^{n\times n}$ which is the induced matrix norm of the Euclidian vector norm. If $A \in R^{n\times n}$ is any given matrix and $\mathcal{M} \subseteq \mathcal{R}^{\backslash\times\backslash}$ is a nonempty set of matrices, we demote by $dist(A, \mathcal{M}) := \inf_{\mathcal{B}\in\mathcal{M}} \|\mathcal{A} - \mathcal{B}\|$ the distance between A and $\mathcal{M}$.

The remainder of the paper is organized as follows: In the next section, the mathematical background and some preliminary results are summarized. The algorithm is proposed in detail in section 3. Section 4 is devoted to proving global local superlinear convergence of the algorithm. Numerical results are reported in section 5.

2. Preliminaries

In this section, we first introduce the conception of NCP-function. A function $\phi : R^2 \to R$ is called an NCP-function if $\phi(a, b) = 0$ is equivalent to $a \geq 0,\ b \geq 0,\ ab = 0$. Let us define the function $H(x) = (h_1(x), h_2(x), \cdots, h_n(x))^T$, where for each $i = 1, 2, \cdots, n$,

$$h_i(x) = \min\{x_i, F_i(x)\}. \tag{2.1}$$

Then NCP (F) can be reformulated as the following nonsmooth equation:

$$H(x) = 0. \tag{2.2}$$

Function h_i and hence H are not differentiable everywhere but semismooth in the sense of Mifflin [17] and Qi [11] if F is continuously differentiable. Denote

$$\alpha(x) \ = \ \{i : F_i(x) < x_i\}, \beta(x) = \{i : F_i(x) = x_i\}, \gamma(x) = \{i : F_i(x) > x_i\}.$$

Then we have

$$h_i(x) = \begin{cases} F_i(x), & \text{if } i \in \alpha(x) \\ \min\{x_i, F_i(x)\}, & \text{if } i \in \beta(x) \\ x_i, & \text{if } i \in \gamma(x) \end{cases}$$

By using the chain rule for generalized derivatives of Lipschitz functions(see [12]), we have the following expression of $\partial_C \Phi(x) = \partial h_1(x) \times \partial h_2(x) \times \cdots \times \partial h_n(x)$ for each $i = 1, 2, \cdots, n$,

$$\partial h_i(x) = \begin{cases} \{\nabla F_i(x)\}, & \text{if } i \in \alpha(x) \\ \{\frac{1}{2}(1+\rho)e_i, \frac{1}{2}(1-\rho)\nabla F_i(x)\}, & \text{if } i \in \beta(x) \\ \{e_i\}, & \text{if } i \in \gamma(x) \end{cases} \tag{2.3}$$