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A NEW MULTI-SYMPLECTIC SCHEME FOR NONLINEAR "GOOD" BOUSSINESQ EQUATION *1)

Lang-yang Huang   Wen-ping Zeng
(*Department of Mathematics, Huaqiao University, Quanzhou 362011, China*)

Meng-zhao Qin
(*LSEC, ICMSEC, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

The Hamiltonian formulations of the linear "good" Boussinesq (L.G.B.) equation and the multi-symplectic formulation of the nonlinear "good" Boussinesq (N.G.B.) equation are considered. For the multi-symplectic formulation, a new fifteen-point difference scheme which is equivalent to the multi-symplectic Preissmann integrator is derived. We also present numerical experiments, which show that the symplectic and multi-symplectic schemes have excellent long-time numerical behavior.

Key words: Nonlinear "good" Boussinesq equation, Multi-symplectic scheme, Preissmann integrator, Conservation law.

1. Introduction

In recent years a remarkable development has taken place in the study of nonlinear evolutionary partial differential equations. An example is the nonlinear "good" Boussinesq (N.G.B.) equation

$$u_{tt} = -u_{xxxx} + u_{xx} + (u^2)_{xx} \tag{1}$$

which describes shallow water waves propagating in both directions. The analytic expression of such solutions is

$$u(x,t) = -A sech^2[(P/2)(\xi - \xi_0)], \qquad \xi = x - ct; \tag{2}$$

where ξ_0 and $P > 0$ are free real parameters and the amplitude A and velocity c of the wave are related to P through the formulas

$$A = 3P^2/2, \qquad c = \pm\sqrt{1-P^2} \tag{3}$$

Note that ξ_0 determines the initial position of the wave, and that, due to the square root in (3), the parameter P can only take values in $0 < P \leq 1$. Thus, the solitary waves (2) only exist for a finite range of velocities $-1 < c < 1$. Of course, a positive (respectively negative) velocity corresponds to a wave moving to the right (respectively to the left). From the available literature we find that for the Korteweg-de Vries(KdV) or cubic schrödinger (CS) equations, the literature is very large, while the study of the N.G.B. equation is only beginning [1, 8, 9, 10].

Hamiltonian systems are canonical systems on phase space endowed with symplectic structures. The dynamical evolutions, i.e., the phase flow of the Hamiltonian systems are symplectic transformations that are area-preserving. The importance of the Hamiltonian systems and their

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special property require the numerical algorithms for them should preserve as much as possible the relevant symplectic properties of the original systems.

Feng Kang[15]–[17] proposed in 1984 a new approach to computing Hamiltonian systems from the view point of symplectic geometry. He systematically described the general method for constructing symplectic schemes with any order accuracy via generating functions. A generalization of the above theory and methods for canonical Hamiltonian equations in infinite dimension can be found in paper [18].

The L.G.B. equation (10) can be written as three infinite dimension Hamiltonian systems. Therefore, it is natural to require a discretization or a semi-discretization to reflect this property. The basic idea is to find a finite dimensional spatial truncation of (10) so that the resulting semi-discretization equation can be cast into a finite dimensional Hamiltonian system. Next, we can integrate the finite dimensional Hamiltonian system in time using a symplectic discretization [11].

However, there are limitations in this approach to developing a symplectic method for PDEs. The disadvantage of this approach is that it is global. To overcome this limitation, Bridges and Reich introduced the concept of multi-symplectic integrators based on a multi-symplectic structure of some conservative PDEs [3, 4]. The theoretical results indicated [4] that the nice features of the multi-symplctic structure are that it is a strictly local concept and that it can be formulated as a conservation law involving differential two forms. Thus the multi-symplectic integrators have excellent local invariant conserving properties. The N.G.B. equation has multi-symplectic structures, therefore we can apply this approach to obtain multi-symplectic integrators.

The purpose of this paper is to present symplectic integrators based on the Hamiltonian formulations of (10) and multi-symplectic integrator based on the multi-symplectic formulation of (1). The outline of this paper is as follows. In section two, we derive the multi-symplectic formulation of the N.G.B. equation and obtain a new fifteen-point multi-symplectic scheme. In section three, we give out three Hamiltonian formulations of the L.G.B. equation (10) with periodic boundary condition and use the hyperbolic function $\tanh(x)$ to construct symplectic schemes of arbitrary order for them. Numerical experiments are presented in section four.

2. Multi-symplectic Formulation of N.G.B. Equation and Multi-symplectic Integrator

We first present the concept of multi-symplectic integrators introduced by Bridges and Reich in [3, 4]. A large class of PDEs (for simplicity,we only consider one space dimension) can be reformulated as a system of the form

$$M\mathbf{z}_t + K\mathbf{z}_x = \bigtriangledown_{\mathbf{z}} S(\mathbf{z}), \mathbf{z} \in \mathbf{R}^n, (x,t) \in \mathbf{R}^2, \tag{4}$$

where M and K are skew-symmetric matrices on $\mathbf{R}^n, n \geq 3$ and $S : \mathbf{R}^n \to \mathbf{R}$ is a smooth function. We call the above system a multi-symplectic Hamiltonian system on a multi-symplectic structure, since it has a multi-symplectic conversation law

$$\frac{\partial}{\partial t}\omega + \frac{\partial}{\partial x}\kappa = 0 \tag{5}$$

where ω and κ are the pre-symplectic forms

$$\omega = \frac{1}{2} dz \wedge M dz \quad \text{and} \quad \kappa = \frac{1}{2} dz \wedge K dz$$

The most significant aspect of the multi-symplectic formulation (4) is that its multi-symplecticity is completely local, which characterizes the system more deeply.

Multi-symplecticity is a geometric property of the PDEs, and we naturally require a discretization to reflect this property. Based on this idea, Bridges and Reich introduced the