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DELAY-DEPENDENT TREATMENT OF LINEAR MULTISTEP METHODS FOR NEUTRAL DELAY DIFFERENTIAL EQUATIONS *1)

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Abstract

This paper deals with a delay-dependent treatment of linear multistep methods for neutral delay differential equations $y'(t) = ay(t) + by(t-\tau) + cy'(t-\tau),\ t > 0,\ y(t) = g(t),\ -\tau \le t \le 0,\ a, b$ and$c \in \mathbb{R}$. The necessary condition for linear multistep methods to be $N\tau(0)$-stable is given. It is shown that the trapezoidal rule is $N\tau(0)$-compatible. Figures of stability region for some linear multistep methods are depicted.

Key words: Delay-dependent stability, Linear multistep methods, Neutral delay differential equations.

1. Introduction

The stability analysis for delay differential equations can be classified into two different categories, i.e. delay-independent and delay-dependent. In the former criterion the stability analysis is carried out for all delay, but in the delay-dependent criterion stability analysis is carried out for arbitrary but fixed delay. The delay-independent analysis was studied by many researches (see e.g. [2, 3, 5]).

Consider the following neutral delay differential equations (NDDEs).

$$\begin{aligned} &y'(t) = ay(t) + by(t-\tau) + cy'(t-\tau), \quad t > 0, \\ &y(t) = g(t), t \in [-\tau, 0]. \end{aligned} \tag{1.1}$$

For $a,\ b$ and $c \in \mathbb{C}$, Bellen *et al.* [2] proved that if:

$$|a\overline{c} - \overline{b}| + |ac + b| < -2\Re[a]. \tag{1.2}$$

Then every solution of (1.1) tends to zero as $t \to \infty$ for all delay. If $a,\ b$ and $c \in \mathbb{R}$ then the condition (1.2) is equivalent to the following condition:

$$|b| < -a \text{ and } |c| < 1, \tag{1.3}$$

which is given by Brayton *et.al.* [5] and is shown in Fig.1.

In the delay-dependent case the delay term also plays a role in the stability analysis. The delay-dependent analysis was first carried out by Al-Mutib [1], but his analysis was based on some numerical experiments only. Recently Guglielmi and Hairer [9, 10, 11] did some work on the delay dependent stability analysis of $\Theta-$ and Runge-Kutta methods for DDEs. In [10] it was proved that linear Θ-methods are $\tau(0)$-stable if and only if they are A-stable. In [13] it was proved that BDF method of second order is $\tau(0)$-stable. Sidibe and Liu [15] proved that all Gauss methods are $N\tau(0)$-stable. In this work we address the delay-dependent stability analysis of linear multistep methods when they are applied to the neutral delay differential equation with real coefficients.

For the sake of simplicity and without losing generality, we consider (1.1) with $\tau = 1$,

$$\begin{aligned} &y'(t) = ay(t) + by(t-1) + cy'(t-1), \quad t > 0, \\ &y(t) = g(t), \quad t \in [-1, 0], \end{aligned} \tag{1.4}$$

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where $a,\ b$ and $c \in \mathbb{R}$.

In this paper we study the stability region of linear multistep methods applied to (1.4) with an arbitrary but fixed value of τ.

The organization of this paper is as follows. In the section 2, the analytical stability region is studied. In section 3, an introduction of linear multistep methods is given and then they are applied to linear test equation (1.4). In section 4, a necessary condition for $N\tau(0)$ -stability is provided and applied to some well-known implicit linear multistep methods for $N\tau(0)$-stability. The results are presented in tabular form (See Table 1). In the section 4, conclusions are presented.

2. Analytical Stability Region

Classically the analytic solution of (1.4) can be expressed by a power series.

$$y(t) = \sum_k \left(A_k \exp(\lambda_k t) + B_k t\ \exp(\lambda_k t)\right),$$

where the coefficients $A_k, B_k \in \mathbb{C}$ are determined by the provided initial function and $\{\lambda_k\}_{k=0}^{\infty}$ are the roots of the quasi-polynomial characteristic equation:

$$\lambda = a + be^{-\lambda} + c\lambda e^{-\lambda}. \tag{2.1}$$

It is well known that the sufficient condition for the asymptotic stability of $y(t)$, independent of the initial function $g(t)$ is,

$$\Re[\lambda_k] < 0, \tag{2.2}$$

for all k.

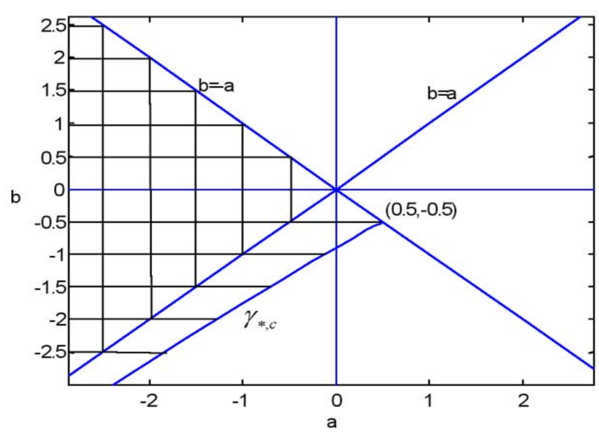

Figure 1: Analytical stability region of (1.4) for $c = 0.5$.

For the case $a,\ b$ and $c \in\ \mathbb{R}$, the stability region Σ_* of (1.4) is given by the connected domain included in the half-plane $a < 1 - c$, and bounded by the planes $|c| = 1$, the straight half-plane l_* and the transcendental surface γ_*. Denoting $\partial\Sigma_*$ as the boundary of the region Σ_*, then

$$\partial\Sigma_* = l_* \cup \gamma_*, \tag{2.3}$$

where

$$l_* = \left\{(a,b,c) \in \mathbb{R}^3 |\ a = -b, a \in (-\infty, 1-c] \text{ and } |c| < 1\right\},$$

$$\gamma_* = \left\{(a,b,c) \in \mathbb{R}^3 |\ c^2\theta^2 + b^2 = a^2 + \theta^2, \theta = \operatorname{arc\,cot} \frac{ab - c\theta^2}{\theta\,(b + ac)},\ \theta \in \mathbb{R}\right\}.$$