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CONSERVATION OF THREE-POINT COMPACT SCHEMES ON SINGLE AND MULTIBLOCK PATCHED GRIDS FOR HYPERBOLIC PROBLEMS*1)

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Abstract

For nonlinear hyperbolic problems, conservation of the numerical scheme is important for convergence to the correct weak solutions. In this paper the conservation of the well-known compact scheme up to fourth order of accuracy on a single and uniform grid is studied, and a conservative interface treatment is derived for compact schemes on patched grids. For a pure initial value problem, the compact scheme is shown to be equivalent to a scheme in the usual conservative form. For the case of a mixed initial boundary value problem, the compact scheme is conservative only if the rounding errors are small enough. For a patched grid interface, a conservative interface condition useful for mesh refinement and for parallel computation is derived and its order of local accuracy is analyzed.

Key words: Conservation, Compact scheme, Uniform grid, Multiblock patched grid.

1. Introduction

In recent years, the compact finite difference method receives an increasing interest due to its high order of accuracy. The compact finite difference method was developed in 1970s within a variety of frameworks. It was then realized that all these methods can be constructed in a systematic way [17]. In [21], a detailed exposition of compact schemes and derivation techniques was given. The history of the development of compact schemes including some works in 1980s is also briefly reviewed in [12]. Recent works in this direction were emphasized in discretization on nonuniform grid [8, 9, 19] or on non-staggered grids[20], method with spectral-like resolution [14], stability of initial-boundary value problem [2, 10] and optimal accuracy for a given grid and initial data [11], parallel treatment of compact schemes[18], control of the group velocity [7], nonlinear compact schemes[4, 5, 6], and mixing with other methods [3].

The main feature of the compact scheme is that the space derivative of the differential equation is computed implicitly and that it is not written in the usual conservative form. For nonlinear hyperbolic problem, conservation of the numerical scheme is required to ensure the solution to converge to a weak solution for vanishing mesh sizes[13]. A compact scheme is however not in the usual conservation form. An important and fundamental question is whether the compact scheme yields numerical solutions which, when the solution converges, converge to weak solutions for vanishing mesh size.

A second important issue is that compact schemes are mainly applied to compute flows in simple geometries with a structured grid. It is very hard to find study of compact schemes for complex geometries. A natural way to apply the compact schemes to complex geometries is obviously by domain decomposition. However, applying domain decomposition to compact schemes is not simple since the compact schemes are inherently implicit. It is not clear how

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to construct interface conditions to ensure independent (and hence parallel) solution of the compact schemes in each subdomain. Besides, for nonlinear hyperbolic problems, conservation at grid interfaces is very important [1, 24].

In this paper we will address these two important issues. First, we want to establish the equivalence between a compact scheme and a usual scheme in conservative form. The usual conservative scheme expresses the increment of the numerical solution at each grid point as the difference between two adjacent numerical fluxes. According to Lax-Wendroff [13], if the numerical flux is consistent with the exact flux function of the hyperbolic system and if the numerical flux involves a finite number of grid points, then the numerical solution is a weak solution if it converges boundedly almost everywhere to some function for vanishing mesh size. However, we will see that the compact scheme, when made equivalent to a scheme in usual conservative form, has a numerical flux involving an infinite number of grid points. Hence we have to extend the convergence theorem of Lax and Wendroff to such a case. This will be done in Section 2.

In Section 3 we will establish the equivalence between a compact scheme and a scheme in usual conservative form. Both initial value problem and initial-boundary-value problems will be considered. For the case of initial-boundary-value problem, it is interesting to note that rounding errors play an important role, especially in the case of a shock wave.

In Section 4, we construct interface conditions for patched grids with and without grid continuity. This is important for mesh refinement and for treating complex geometries. The interface treatment is required to be conservative and accurate, and to ensure independent or parallel solution of the implicit schemes in each subdomain.

In Appendix A, we show that it is not evident that compact schemes with non-constant coefficients are conservative.

2. Conservation for a Nonlinear Hyperbolic Equation

2.1. Hyperbolic Equation and Numerical Solution

One of the great advantage of the compact scheme is that each space direction can be treated independently of the others. Hence one can just consider a one-dimensional problem. The case of multidimensions for patched grid will be considered in the end of this paper.

Let us consider the scalar hyperbolic equation

$$u_t + f(u)_x = 0, \;\; x \in \Omega, \;\; t > 0 \tag{2.1}$$

together with the initial condition

$$u(x, 0) = u_0(x), \;\; x \in \Omega \tag{2.2}$$

If $\Omega = \mathbb{R}$, then (2.1)-(2.2) define a pure initial-value problem. If $\Omega = (-1, 1)$, then (2.1)-(2.2) together with the boundary condition

$$u(-1, t) = g(t) \;\; \text{if} \;\; a(u) = f'(u) > 0 \tag{2.3}$$

define a mixed initial-boundary-value problem.

The result derived under the assumption of a scalar equation remains valid for a system of hyperbolic equations, since the compact scheme is component-invariant. When we treat problems with shock waves, an upwinding compact scheme is necessary, which can be simply done, in the case of a system, by splitting the flux f into a positive part f^+ and a negative part f^- according to the positive eigenvalues and negative eigenvalues of the Jacobian matrix $A =. \partial_u f(u)$. Then the positive part f^+ is approximated through a left-sided compact scheme, and the negative part f^- is approximated through a right-sided compact scheme. See [7] for