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MATHEMATICAL ANALYSIS FOR QUADRILATERAL ROTATED $\mathcal{Q}_1$ ELEMENT II: POINCARÈ INEQUALITY AND TRACE INEQUALITY*[1)]

Ping-bing Ming   Zhong-ci Shi
(*Institute of Computational Mathematics, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract

This is the second part of the paper for the mathematical study of nonconforming rotated $\mathcal{Q}_1$ element (NRQ$_1$ hereafter) on arbitrary quadrilateral meshes. Some *Poincarè Inequalities* are proved *without* assuming the quasi-uniformity of the mesh subdivision. A discrete trace inequality is also proved.

Key words: Quadrilateral rotated $\mathcal{Q}_1$ element, Poincarè inequality, Trace inequality.

1. Mesh Subdivision

Let $\mathcal{T}_h$ be a partition of $\bar{\Omega}$ by convex quadrilaterals K with the mesh size h_K and $h := \max_{K\in\mathcal{T}_h} h_K$. We assume that $\mathcal{T}_h$ is shape regular in the sense of Ciarlet-Raviart [3, p. 247]. We define a mesh condition which actually quantifies the deviation of a quadrilateral away from a parallelogram [10].

Definition 1.1. (1+α)-Section Condition ($0 \le \alpha \le 1$). *The distance d_K between the midpoints of two diagonals of $K \in \mathcal{T}_h$ is of $\mathcal{O}(h_K^{1+\alpha})$ uniformly for all elements K as $h \to 0$. In case of $\alpha = 0$, $\mathcal{T}_h$ is the trapezoid mesh, and in case of $\alpha = 1$, $\mathcal{T}_h$ satisfies the* Bi-Section Condition *[13].*

We define by $\mathcal{P}_k$, the space of polynomials of degrees no more than k, and by $\mathcal{Q}_k$, the space of degrees no more than k in each variable.

Let $\hat{K} = [-1,1]^2$ be the reference square, the coordinates of the its four vertices are denoted by $\{(\xi_i,\eta_i)\}_{i=1}^4$ which is labelled from the lower-left to the upper-left in a counterclockwise manner, the same rule applies to K, whose vertices are denoted by $\{(x_i,y_i)\}_{i=1}^4$. There exists a bilinear mapping $\boldsymbol{F}$ such that $\boldsymbol{F}(\hat{K}) = K$. Let $\boldsymbol{F} = (x^K, y^K)$, with

$$x^K := \frac{1}{4}\sum_{i=1}^4 (1+\xi_i\xi)(1+\eta_i\eta)x_i = a_0 + a_1\xi + a_2\eta + a_{12}\xi\eta,$$
$$y^K := \frac{1}{4}\sum_{i=1}^4 (1+\xi_i\xi)(1+\eta_i\eta)y_i = b_0 + b_1\xi + b_2\eta + b_{12}\xi\eta.$$

To each scalar function $\hat{v}$ defined on $\hat{K}$, we associate it a function v on K such that $v(\boldsymbol{x}) = v(\boldsymbol{F}(\hat{\boldsymbol{x}})) = \hat{v}(\hat{\boldsymbol{x}})$.

Before closing this section, we fix some notations. For any integer k, $H^k(\Omega)$ denotes the standard Sobolev spaces [5]. $⨍_\Omega u\,dx$ is defined as the integral average of u on Ω. Denote by V_h the NRQ$_1$ finite element space, and by V_h^a, V_h^p the corresponding finite element spaces

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with continuous edge integral mean or with continuous mid-point on each edge (see [12,9] for a definition). For any $v \in V_h$, we define a piecewise norm as

$$|v|_h := \Big(\sum_{K \in \mathcal{T}_h} \|\nabla v\|_{0,K}^2 \Big)^{1/2}.$$

Throughout this paper, the generic constant C is assumed to be independent of the mesh size h.

2. Poincarè Inequality

In this section, we present some versions of the *Poincaré inequality* for the nonconforming finite element spaces V_h^a and V_h^p [12]. We adopt all the notations appeared in [9].

In case the element K is a rectangular parallelepiped, the *Poincarè inequality* has been proved in [6]. A strengthened version of this inequality is presented in [7]. But both of them are suitable only for the homogeneous space $V_{0,h}$. Moreover, as to the strengthened *Poincarè Inequality*, the quasi-uniformity of the mesh subdivision is assumed. In this section, we will extend *Poincaré inequalities* appeared in [6, 7] to both homogeneous and nonhomogeneous spaces over arbitrary quadrilateral meshes *without* the quasi-uniformity assumption, which allows for the adaptive mesh subdivision.

Meanwhile, some generalized *Poincarè Inequalities* have been proved by Stummel in [14] by virtue of the compact argument. However, when it applies to the quadrilateral rotated $\mathcal{Q}_1$ element, we have to assume the quasi-uniformity of meshes and the closedness of the given finite element space via the generalized patch test. But as we have seen in [9] that the finite element space V_h^p does not pass the generalized patch test for arbitrary quadrilaterals. So, instead of the compact argument, we adopt Teman's approach [15] which avoids the generalized patch test.

There is also another approach appeared in [8, Chp.3] to prove the *Poincaré inequality* for nonconforming elements, which starts from the conforming "relative" of the relevant nonconforming element, then exploits the high order distance between the conforming "relative" and the nonconforming element to prove the desired inequality. This approach is very flexible which allows for very "rough" mesh. Recently the same approach is employed by Brenner [2] to prove the generalized *Poincaré-Friedrichs inequality* for piecewise H^1 functions.

Theorem 2.1. *Poincarè Inequality*

$$\|v\|_0 \le C|v|_h \qquad \forall v \in V_{0,h}. \tag{2.1}$$

$$\|v\|_0 \le C\Big(|v|_h + |\fint_\Omega v\,dx|\Big) \qquad \forall v \in V_h. \tag{2.2}$$

$$\|v\|_0 \le C(|v|_h + \|v\|_{0,\Gamma}) \qquad \forall v \in V_h^a. \tag{2.3}$$

$$\|v\|_0 \le C|v|_h + C\Big(\sum_{e \subset \Gamma \cap \mathcal{T}_h, M \in \mathcal{F}} |e||v(\mathcal{M})|^2 \Big)^{1/2} \qquad \forall v \in V_h^p. \tag{2.4}$$

Proof. We prove the above four inequalities one by one.

For any $\boldsymbol{\psi} \in [\mathcal{H}^1(\Omega)]^2$ and $v \in V_{0,h}$, an integration by parts gives

$$\int_\Omega \operatorname{div} \boldsymbol{\psi} v\,dx = \sum_{K \in \mathcal{T}_h} \Big(-\int_K \boldsymbol{\psi} \nabla v\,dx + \int_{\partial K} v \boldsymbol{\psi} \cdot \boldsymbol{n}\,ds \Big).$$