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LEAST-SQUARES SOLUTION OF $AXB = D$ OVER SYMMETRIC POSITIVE SEMIDEFINITE MATRICES X [*1)]

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Abstract

Least-squares solution of $AXB = D$ with respect to symmetric positive semidefinite matrix X is considered. By making use of the generalized singular value decomposition, we derive general analytic formulas, and present necessary and sufficient conditions for guaranteeing the existence of the solution. By applying MATLAB 5.2, we give some numerical examples to show the feasibility and accuracy of this construction technique in the finite precision arithmetic.

Key words: Least-squares solution, Matrix equation, Symmetric positive semidefinite matrix, Generalized singular value decomposition.

1. Introduction

Denote by $R^{n\times m}$ the set of all real $n \times m$ matrices, I_k the identity matrix in $R^{k\times k}$, $\mathrm{O}R^{n\times n}$ the set of all orthogonal matrices in $R^{n\times n}$, $\mathrm{S}R^{n\times n}$ the set of all symmetric matrices in $R^{n\times n}$, and $\mathrm{S}R_0^{n\times n}(\mathrm{S}R_+^{n\times n})$ the set of all symmetric positive semidefinite (definite) matrices in $R^{n\times n}$. The notations $A \geq O$ $(A > O)$ represents that the matrix A is a symmetric positive semidefinite (definite) matrix, A^+ represents the Moor-Penrose generalized inverse of a matrix A, and $\|\cdot\|$ denotes the Frobenius norm. We use both O and 0 to denote the zero matrix.

The purpose of this paper is to study the least-squares problem of the matrix equation $AXB = D$ with respect to $X \in \mathrm{S}R_0^{n\times n}$, i.e.,

Problem I. For given matrices $A \in R^{m\times n}$, $B \in R^{n\times p}$, $D \in R^{m\times p}$, find a matrix $\hat{X} \in \mathrm{S}R_0^{n\times n}$ such that

$$\|A\hat{X}B - D\| = \min_{X\in \mathrm{S}R_0^{n\times n}} \|AXB - D\|.$$

Allwright[1] first investigated a special case of Problem I, i.e.,

Problem A. For given matrices $B, D \in R^{n\times m}$, find a matrix $\hat{X} \in \mathrm{S}R_0^{n\times n}$ such that

$$\|\hat{X}B - D\| = \min_{X\in \mathrm{S}R_0^{n\times n}} \|XB - D\|.$$

The solution of Problem A can be used as estimates of the inverse Hessian of a nonlinear differentiable function $f : R^n \to R^1$, which is to be minimized with respect to a parameter

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vector $x \in R^n$ by a quasi-Newton-type algorithm. Allwright, Woodgate[2,3] and Liao[4] gave some necessary and sufficient conditions for the existence of solution of Problem A as well as explicit formulas of the solution for some special cases.

Dai and Lancaster[5] studied in detail another special case of Problem I, i.e.,

Problem B. For given matrices $A \in R^{n\times m}$, $D \in R^{m\times m}$, find a matrix $\hat{X} \in \mathrm{S}R_0^{n\times n}$ such that

$$\|A^T\hat{X}A - D\| = \min_{X\in \mathrm{S}R_0^{n\times n}} \|A^TXA - D\|.$$

An inverse problem[6,7] arising in structural modification of the dynamic behaviour of a structure calls for solution of Problem B. Dai and Lancaster[5] successfully solved Problem B by using singular value decomposition(SVD).

Obviously, Problem I is a nontrivial generalization of Problems A and B, and the approachs adopted for solving Problems A and B in [1-5] are not suitable for Problem I.

In this paper, by applying the generalized singular value decomposition (GSVD) we will present necessary and sufficient conditions for the existence of the solution of Problem I, and give analytic expression of it, too.

2. Solutions of Problem I

We first study the solution of Problem I when matrices $A, B, D \in R^{n\times n}$ and $\mathrm{rank}(A) = rank(B) = n$.

Lemma 2.1[1]**.** *If rank*$(B) = n$*, then Problem A has a unique solution.*

Theorem 2.1. *If* $A, B, D \in R^{n\times n}$ *and rank*$(A) =$ *rank*$(B) = n$*, then Problem I has a unique solution.*

Proof. Because $A, B \in R^{n\times n}$ and $\mathrm{rank}(A) = \mathrm{rank}(B) = n$, we easily know that $\mathrm{rank}(\bar{B}) = n$ and

$$\|AXB - D\| = \|(AXA^T)(A^{-T}B) - D\| = \|\bar{X}\bar{B} - D\|, \tag{2.1}$$

where $\bar{B} = A^{-T}B$, and $\bar{X} = AXA^T$. Now, it is obvious that $\bar{X} \geq 0$ if and only if $X \geq 0$. Therefore, by Lemma 2.1 and (2.1) Problem I has a unique solution.

To study the solvability of Problem I in general case, we decompose the given matrix pair $[A^T, B]$ by GSVD[8] as follows:

$$A^T = M\Sigma_A U^T, \qquad B = M\Sigma_B V^T, \tag{2.2}$$

where M is an $n\times n$ nonsingular matrix, and

$$\Sigma_A = \begin{array}{cc} \begin{pmatrix} I_A & O & O \\ O & S_A & O \\ O & O & O_A \\ O & O & O \end{pmatrix} & \begin{array}{c} r \\ s \\ k-r-s \\ n-k \end{array} \\ \begin{array}{ccc} r & s & m-r-s \end{array} & \end{array}, \qquad \Sigma_B = \begin{array}{cc} \begin{pmatrix} O_B & O & O \\ O & S_B & O \\ O & O & I_B \\ O & O & O \end{pmatrix} & \begin{array}{c} r \\ s \\ k-r-s \\ n-k \end{array} \\ \begin{array}{ccc} p+k-r & s & k-r-s \end{array} & \end{array},$$

$$k = \mathrm{rank}(A^T, B), \quad r = k - \mathrm{rank}(B), \quad s = \mathrm{rank}(A^T) + \mathrm{rank}(B) - k,$$

I_A and I_B are identity matrices, O_A and O_B are zero matrices, and

$$S_A = \mathrm{diag}(\alpha_1, \alpha_2, \ldots, \alpha_s), \quad S_B = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_s)$$

with

$$1 > \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_s > 0, \quad 0 < \beta_1 \leq \beta_2 \leq \ldots \leq \beta_s < 1, \quad \alpha_i^2 + \beta_i^2 = 1 (i = 1, 2, \ldots, s),$$

$$U = (\underset{r}{U_1}, \; \underset{s}{U_2}, \; \underset{m-r-s}{U_3}) \in \mathrm{O}R^{m\times m}, \quad V = (\underset{p+r-k}{V_1}, \; \underset{s}{V_2}, \; \underset{k-r-s}{V_3}) \in \mathrm{O}R^{p\times p}.$$