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EXPLICIT BOUNDS OF EIGENVALUES FOR STIFFNESS MATRICES BY QUADRATIC HIERARCHICAL BASIS METHOD[*1)]

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Abstract

The bounds for the eigenvalues of the stiffness matrices in the finite element discretization corresponding to $Lu := -u''$ with zero boundary conditions by quadratic hierarchical basis are shown explicitly. The condition number of the resulting system behaves like $O(\frac{1}{h})$ where h is the mesh size. We also analyze a main diagonal preconditioner of the stiffness matrix which reduces the condition number of the preconditioned system to $O(1)$.

Key words: hierarchical basis, multilevel

1. Introduction

The main object in this paper is to investigate the explicit bounds of the eigenvalues for the stiffness matrix B_j arisen from the finite element method using piecewise quadratic hierarchical multilevel basis instead of the usual piecewise quadratic nodal basis for the one dimensional elliptic operator $Lu := -u''$ with zero boundary conditions defined on $[0,1]$. Hence the condition number of B_j can be shown as about $\frac{12}{h}$. One may use an interpolation operator to obtain the asymptotic behavior of condition numbers like $O(\frac{1}{h})$, but such technique does not yield the explicit bound for the eigenvalues (see [5]).

For piecewise linear hierarchical multilevel elements, the condition numbers are analyzed in [4, 6, 7]. One can easily see that the stiffness matrix for the unidimensional case using piecewise linear hierarchical basis becomes a diagonal matrix. This phenomenon is quite different from the case of piecewise quadratic hierarchical multilevel elements mainly because of their non-orthogonalities in the H^1 sense.

Define the bilinear form corresponding to $Lu = -u''$ with zero boundary conditions as

$$b(u,v) = \int_0^1 u'v'dx. \tag{1.1}$$

Following the ideas [6], we will give the upper bound $\frac{16(\sqrt{2}+1)}{3(\sqrt{2}-1)\,h}$ of $b(u,u)$ in terms of the Euclidean norm of the hierarchical coefficient vector of u when u is represented by the piecewise quadratic hierarchical basis where h denotes the uniform mesh size of the final level space. In order to get the uniform lower bound $\frac{8}{3}$ of $b(u,u)$, we will use a lower bound for eigenvalues of certain symmetric matrices which will be given in Appendix. There are lots of literature on the

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multilevel or two level hierarchical bases. One may refer to, for example, [1, 2, 3, 4, 5, 6, 7] for understanding of them.

The rest of paper is as follows. In section 2, we provide some of definitions and preliminaries on the piecewise quadratic hierarchical basis. In section 3, we analyze the explicit bounds of the stiffness matrix B_j using the ideas given in [6] and the results given in Appendix. We analyze a preconditioner which is the main diagonal of the stiffness matrix and give a numerical experiment in section 4. Appendix which plays an important role to investigate the result in section 3 will be provided in the last of this paper.

2. Preliminaries

For $I = [0,1]$, let us denote π a uniform partition of I, that is to say, any elements of π has same length such that the union of these intervals is I and such that the intersection of two subintervals of π either consists of a common knot of both intervals or is empty. Let $\pi_0 = I$ be a coarse initial partition, beginning with this partition we construct a nested family $\pi_0, \pi_1, \pi_2, \cdots$ of partitions of I where π_{k+1} is obtained from π_k by subdividing each interval of π_k into two subintervals having the same size. Note that the partition π_k of level k has the mesh size $h_k := (1/2)^k$.

Denote by $\|\cdot\|_{0,D}$ the usual $L^2(D)$-norm, $\|\cdot\|_{1,D}$ the usual Sobolev $H^1(D)$-norm and $|\cdot|_{1,D}$ the Sobolev $H^1(D)$-seminorm. Throughout this paper, we will use k and ℓ for levels, j for last level, p and q denote indices for nodes.

Let $\mathcal{N}_k (k = 0, 1, \cdots, j)$ be the set consisting of the nodes of the intervals of π_k, their mid-points and end-points of I and let $\mathcal{S}_k$ be the subspace of the Sobolev space $H_0^1(I)$ which consists of all continuous functions on I that is quadratic on the intervals of π_k vanishing at two boundary points 0 and 1. We call the function in $\mathcal{S}_k$ finite element functions of level k. Obviously we have $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ from the fact that $\mathcal{N}_k \subset \mathcal{N}_{k+1}$, and a function $u \in \mathcal{S}_k$ is determined by its values at the nodes $x \in \mathcal{N}_k$.

Let J_k be the interpolation operator from $\mathcal{S}_j$ to $\mathcal{S}_k$ such that

$$J_k u \in \mathcal{S}_k; \quad (J_k u)(x) = u(x), \quad x \in \mathcal{N}_k.$$

For $k \le j$, a function $u \in \mathcal{S}_k$ can be reproduced by the interpolation operator J_j, so that any function $u \in \mathcal{S}_j$ has the representation

$$u = J_0 u + \sum_{k=1}^{j} (J_k u - J_{k-1} u), \tag{2.1}$$

which is a decomposition of u into fast oscillating functions corresponding to the different refinement levels. Note that $J_0 u$ is a function of the finite element space corresponding to the initial partition and $J_k u - J_{k-1} u \in \mathcal{S}_k$ vanishes at all nodal points of level $k-1$.

Let $\mathcal{V}_k (k = 1, 2, \cdots, j)$ be the subspace of $\mathcal{S}_k$ consisting of all finite element functions vanishing at the nodes of level $k-1$. Let us denote the nodes in $\mathcal{N}_k \setminus \mathcal{N}_{k-1}$ as $\{x_p^k \,|\, p = 1, 2, \cdots, d_k\}$ and let $\mathcal{V}_0 := \mathcal{S}_0$ with $\mathcal{N}_0 := \{x_1^0\}$. Then we can easily check that the dimension of $\mathcal{V}_k$ is $d_k := 2^k$, $\mathcal{V}_k$ is the range of $J_k - J_{k-1}$, and (2.1) means that $\mathcal{S}_j$ is the direct sum of $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \cdots, \mathcal{V}_{j-1}$ and $\mathcal{V}_j$. The hierarchical basis of $\mathcal{S}_k$, $k \ge 1$, consists of the old hierarchical basis functions of level $k-1$ and the functions forming a nodal basis of $\mathcal{V}_k$. For a function $u \in \mathcal{S}_j$ with hierarchical basis of this space, we can represent u as

$$u = \sum_{k=0}^{j} u^k \in \mathcal{S}_j \quad \text{with} \quad u^k = \sum_{p=1}^{d_k} u_p^k \phi_p^k \in \mathcal{V}_k \tag{2.2}$$

where the quadratic hierarchical basis $\{\phi_p^k : k = 0, 1, \cdots, j,\ p := p(k) = 1, 2, \cdots, d_k\ \}$ of $\mathcal{S}_j$ is